

The Brauer-Picard group of the Asaeda-Haagerup fusion categories

PINHAS GROSSMAN
NOAH SNYDER

URLs: <http://math.columbia.edu/~nsnyder>
<http://w3.impa.br/~pinhas>

Email: pinhas@impa.br nsnyder@math.columbia.edu

Abstract We prove that the Brauer-Picard group of Morita autoequivalences of each of the three fusion categories which arise as an even part of the Asaeda-Haagerup subfactor or of its index 2 extension is the Klein four-group. We describe the 36 bimodule categories which occur in the full subgroupoid of the Brauer-Picard groupoid on these three fusion categories. We also classify all irreducible subfactors both of whose even parts are among these categories, of which there are 111 up to isomorphism of the planar algebra (76 up to duality). Although we identify the entire Brauer-Picard group, there may be additional fusion categories in the groupoid. We prove a partial classification of possible additional fusion categories Morita equivalent to the Asaeda-Haagerup fusion categories and make some conjectures about their existence; we hope to address these conjectures in future work.

AMS Classification 46L37; 18D10

Keywords Fusion categories, subfactors, quantum subgroups

1 Introduction

If $N \subset M$ is a finite index subfactor, then the fundamental bimodule ${}_NM_M$ generates tensor categories of $N - N$ and $M - M$ bimodules, called the even parts of the subfactor, as well as a Morita equivalence (i.e. an invertible bimodule category) of $N - M$ bimodules between them. It is natural to ask: what is the full Morita equivalence class of the even parts, and what are all the invertible bimodule categories between tensor categories in this class?

In general, not much can be said about this question. But in the case that $N \subset M$ has finite depth, the even parts are fusion categories, and a result known as Ocneanu rigidity says that the “maximal atlas” [Ocn01] of Morita equivalences is a finite groupoid, called the Brauer-Picard groupoid [ENO10]. (In fact the Brauer-Picard groupoid is a 3-groupoid, but in this paper we only consider the 1-truncation.)

Intimately related to the Brauer-Picard groupoid of a fusion category \mathcal{C} is the family of module categories over \mathcal{C} . Certainly any bimodule category is in particular a module category over each of its left and right arguments, but also to any simple module category ${}_{\mathcal{C}}\mathcal{K}$, there is associated a dual category of module endofunctors \mathcal{D} such that ${}_{\mathcal{C}}\mathcal{K}_{\mathcal{D}}$ is a Morita equivalence.

For the fusion category Rep_G of finite-dimensional complex representations of a finite group G the situation is well understood. Let $H \subseteq \tilde{H}$ be a central extension by \mathbb{C}^* of a subgroup $H \subseteq G$. Then $Rep_{\tilde{H}}$ is a module category over Rep_G , and every simple module category over Rep_G is of this form [Ost03]. For this reason module categories are sometimes called quantum subgroups.

Another class of fusion categories for which the representation theory is known is the family coming from quantum \mathfrak{su}_2 at roots of unity, which is parametrized by the Dynkin diagrams of type A_n . Here the module categories corresponding to Dynkin diagrams of type $A - D - E$ which have the same Coxeter number as A_n (see [Ocn88, Ocn99, BEK00] for this result in subfactor language, and [KO02, Ost03, EO04] for the translation of these results into the language of fusion categories and module categories); the corresponding Morita equivalences are implemented by the Goodman-de la Harpe-Jones subfactors [GdlHJ89]. Ocneanu has also announced the classification of quantum subgroups of the fusion categories coming from quantum \mathfrak{su}_3 and \mathfrak{su}_4 [Ocn02] (see [EP09a, EP09b] for details in the \mathfrak{su}_3 case). However, besides for these families there are very few examples for which a complete classification of quantum subgroups is known.

Motivated by the classification of small-index subfactors, Asaeda and Haagerup constructed two subfactors, one with index $\frac{5+\sqrt{13}}{2}$, known as the Haagerup subfactor, and one with index $\frac{5+\sqrt{17}}{2}$, known as the Asaeda-Haagerup subfactor [AH99]. They called these subfactors “exotic” as they did not appear to be related to any previously known mathematical objects. We call the even parts of these subfactors, as well as any Morita equivalent fusion categories, the Haagerup and Asaeda-Haagerup fusion categories, respectively. A basic question is to determine the representation theory (i.e. the quantum subgroups and the Brauer-Picard groupoid) of these fusion categories.

In [GS11] we considered the Haagerup fusion categories, and in the present paper we consider the Asaeda-Haagerup fusion categories. We now briefly describe the results and compare the two cases.

In the Haagerup case [GS11], we began with the Haagerup subfactor which gave a Morita equivalence between a fusion category \mathcal{H}_1 with commutative fusion rules, and a fusion category \mathcal{H}_2 with noncommutative fusion rules. The category

\mathcal{H}_2 contains an order 3 invertible object α . It turned out that the category of $(1 + \alpha + \alpha^2) - (1 + \alpha + \alpha^2)$ bimodules in \mathcal{H}_2 is a new category (although it has the same fusion rules as \mathcal{H}_2), which we called \mathcal{H}_3 . This ended up being everything: there is a unique Morita equivalence between each not-necessarily-distinct pair of these three categories, and there are no other Morita equivalent categories. In particular, the Brauer-Picard group of Morita autoequivalences of each of the three categories is trivial.

In the Asaeda-Haagerup case, we begin with the Asaeda-Haagerup subfactor, which gives a Morita equivalence between a fusion category with commutative fusion rules, which we call \mathcal{AH}_1 and a fusion category with noncommutative fusion rules, which we call \mathcal{AH}_2 . This time the category \mathcal{AH}_2 contains an order 2 invertible object α , and again the category of $(1 + \alpha) - (1 + \alpha)$ bimodules in \mathcal{AH}_2 is a new category (although this time all three categories have different fusion rules), which we call \mathcal{AH}_3 . However unlike in the Haagerup case, this is not everything.

The key to finding the rest of the groupoid is the existence of several additional small-index subfactors. Motivated by the study of quadrilaterals of factors, it was shown in [GI08] and [AG11] that there are subfactors with indices one larger than the Haagerup and Asaeda-Haagerup subfactors, i.e. $\frac{7+\sqrt{13}}{2}$ and $\frac{7+\sqrt{17}}{2}$; we call these subfactors $H + 1$ and $AH + 1$. In the Haagerup case $H + 1$ just implements the trivial autoequivalence of \mathcal{H}_1 , so it does not give any new information about the groupoid (and indeed, in [GS11] we gave a “trivial” construction of $H + 1$ exploiting this fact). However in the Asaeda-Haagerup case, $AH + 1$ gives a second Morita equivalence between \mathcal{AH}_1 and \mathcal{AH}_3 , which immediately implies that the group of autoequivalences of each of the Asaeda-Haagerup fusion categories is non-trivial.

Moreover, it was conjectured in [AG11] that the “plus one” construction can be iterated once more in the Asaeda-Haagerup case to find a subfactor $AH + 2$ with index $\frac{9+\sqrt{17}}{2}$. We verify the existence of $AH + 2$ and show that it gives a new autoequivalence of \mathcal{AH}_1 which is not in the groupoid generated by AH and $AH + 1$. Finally, we find the full group of Morita autoequivalences of each of the three Asaeda-Haagerup fusion categories:

Theorem 1.1 *The Brauer-Picard group of Morita autoequivalences of each of the Asaeda-Haagerup fusion categories is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.*

This means that there are a total of 36 bimodule categories between the three Asaeda-Haagerup fusion categories. We compute the fusion rules for all of these

bimodule categories, and classify all subfactors which realize them (up to isomorphism of the planar algebra). There are over 100 such subfactors and we compute all of their principal graphs.

Although we are able to identify the full Brauer-Picard group of Morita autoequivalences of these three Asaeda-Haagerup fusion categories, we do not identify the full Brauer-Picard groupoid. In addition to the three fusion categories which we construct, there may be several additional fusion categories Morita equivalent to them. We make several conjectures concerning the full Brauer groupoid which we hope to address in the future. In particular, it appears as though the Asaeda-Haagerup fusion categories are Morita equivalent to several more symmetric fusion categories with 4 invertible objects. We hope that these additional fusion categories could open the door to a more symmetric construction of the Asaeda-Haagerup subfactor, and to generalizations where $\mathbb{Z}/4\mathbb{Z}$ is replaced by other groups.

The nontriviality of the group of autoequivalences allows us to apply the recently developed extension theory of fusion categories of [ENO10]. In a follow-up paper with David Jordan we show that there are no nontrivial invertible objects in the Drinfeld center of the Asaeda-Haagerup categories; the extension theory then implies that there are $\mathbb{Z}/2\mathbb{Z}$ -graded fusion categories associated to each of the nontrivial autoequivalences in the Brauer-Picard groupoid of which the zero-graded parts are the corresponding Asaeda-Haagerup fusion categories. This gives 9 new fusion categories, including one coming from $AH + 2$ which has a self-dual object with the relatively small dimension $\sqrt{\frac{9+\sqrt{17}}{2}} \approx 2.56155281$.

The main technique of this paper is to analyze the combinatorial structure at the level of fusion rings for the whole Brauer-Picard groupoid simultaneously. In particular, instead of considering principal graphs one at a time (as we did in [GS11]) we consider the richer structure of a fusion module (or nimrep), and instead of considering one fusion module at a time we consider the whole structure of all possible fusion bimodules and all possible rules for composition. These techniques allow us to eliminate certain combinatorial possibilities which look fine on their own, but which cannot be made compatible with all the other Morita equivalences.

Two aspects of our analysis required heavy computation. First, verifying the existence of $AH + 2$ requires two difficult computations. We give an alternate construction of $AH + 1$ following the outline of Asaeda-Haagerup's construction of the Asaeda-Haagerup subfactor, and then we construct $AH + 2$ by checking the algebra object relations following Asaeda-Grossman. Conceptually these are very close to the original calculations, but both calculations are more difficult

and we used C++ and Mathematica for bookkeeping. Second, to classify module categories over the Asaeda-Haagerup fusion categories, it was necessary to first classify fusion modules over the corresponding fusion rings. Additionally, we had to check the multiplicative compatibility of triples of fusion modules, i.e. whether there exists a map from the relative tensor product of two fusion modules to a given third module which is compatible with the various fusion ring actions, Frobenius reciprocity criteria, and Frobenius-Perron dimensions. This was done through an elaborate combinatorial search written in C++; we outline the basic ideas of this search in Section 5 below.

The organization of the paper is as follows:

In Section 2, we present some background material on fusion categories, subfactors, and connections which will be necessary for what follows.

In Section 3, we provide diagrammatic proofs of two theorems about algebra objects in fusion categories which we require in later sections.

In Section 4, we describe three Asaeda-Haagerup fusion categories and their relation to the subfactors AH , $AH + 1$, $AH + 2$, and give their Grothendieck rings. We also construct $AH + 2$; since the calculations are tedious and very closely analogous to the arguments in [AH99, AG11] we only give a rapid sketch.

In Section 5, we describe the combinatorial data which you get when you de-categorify the Brauer-Picard groupoid (several fusion rings, fusion bimodules between them, and rules for composition) and the strong compatibility conditions that they must satisfy (like Frobenius reciprocity); we explain how to compute representations of fusion rings; and we present the classification of fusion modules and fusion bimodules over the three Asaeda-Haagerup rings. We also introduce the property of multiplicative compatibility for triples of fusion bimodules and show how to compute all possible multiplications between fusion bimodules.

In Section 6, we find the Brauer-Picard group of the Asaeda-Haagerup fusion categories and describe the full subgroupoid of the Brauer-Picard group generated by AH , $AH + 1$, $AH + 2$. We also give several results about possible additional objects in the Brauer-Picard groupoid and make conjectures about their existence.

In Section 7, we classify all subfactors whose even parts are among these three Asaeda-Haagerup fusion categories.

We also include with the paper the following supplementary data files in plain text, which may be found in the arxiv source: *AH1Modules*, *AH2Modules*,

AH3Modules, *Bimodules*, *BimoduleCompatibility*, and *Module Compatibility*. These files contain the multiplication tables for the fusion modules and bimodules over the Asaeda-Haagerup fusion rings as well as the lists of compatible compositions of modules and bimodules.

Acknowledgements: The authors would like to thank Marta Asaeda for help computing the connection on the fundamental 4-graph in $AH + 1$ and for useful conversations. The authors would like to thank Scott Morrison for pointing out that the dual graph of $AH + 2$ must be the same as the principal graph. The authors would also like to thank Masaki Izumi, David Jordan, and Emily Peters for helpful conversations. The Fusion Atlas was helpful for drawing graphs and checking their Frobenius-Perron weights, although we do not use it in any proofs. Pinhas Grossman was partially supported by NSF grant DMS-0801235, and Noah Snyder was supported by an NSF Postdoctoral Fellowship at Columbia University.

2 Background

2.1 Fusion categories

Definition 2.1 [ENO05, DR89]. A fusion category over a field k (which will always be \mathbb{C} in this paper) is a k -linear semisimple rigid monoidal category with finitely many simple objects up to isomorphism and simple identity object. A conjugation on a \mathbb{C} -linear monoidal category is a contravariant involutive endofunctor with fixes objects and commutes with the tensor product. A C^* -tensor category is a rigid \mathbb{C} -linear semisimple monoidal category with conjugation $*$ such that: (a) every morphism space is a Banach space and (b) $\|f \circ g\| \leq \|f\| \|g\|$ and $\|f^* \circ f\| = \|f^*\| \|f\|$ for all composable morphisms f, g . A unitary fusion category is a fusion category which is also a C^* -tensor category.

Fusion categories are categorifications of rings, and there are corresponding categorified notions of module categories (left and right) and bimodule categories. For definitions, as well as the notion of tensor product of bimodule categories and invertibility, see [Ost03, ENO05]. All module categories are assumed to be semisimple.

Definition 2.2 [Müg03, ENO10]. A Morita equivalence between two fusion categories \mathcal{C}, \mathcal{D} is an invertible $\mathcal{C} - \mathcal{D}$ bimodule category. The Brauer-Picard groupoid of a fusion category \mathcal{C} is the category whose objects are fusion categories which are Morita equivalent to \mathcal{C} and whose morphisms are Morita

equivalences between two such fusion categories (again considered up to equivalence as bimodule categories.)

Remark 2.3 The Brauer-Picard group is actually defined in [ENO10] as a 3-groupoid. In this paper we only consider the 1-truncation.

Every Morita equivalence is indecomposable both as a left and right module category [ENO10]. If ${}_{\mathcal{C}}\mathcal{K}_{\mathcal{D}}$ is a Morita equivalence, then \mathcal{D} is the dual category (i.e. the category of module endofunctors) to the module category ${}_{\mathcal{C}}\mathcal{K}$.

Definition 2.4 An algebra object in a monoidal category is an object A together with morphisms $1 \rightarrow A$ and $A \otimes A \rightarrow A$ satisfying the usual unit and associativity axioms.

Given an algebra object A in a fusion category \mathcal{C} , one can define left and right module objects over A in \mathcal{C} . The category of right A -modules in \mathcal{C} is a left module category over \mathcal{C} ; it is also a right module category over the category of $A - A$ bimodules in \mathcal{C} , where the action is by the relative tensor product over the algebra object A . The resulting bimodule category is a Morita equivalence.

Definition 2.5 [Ost03]. Let ${}_{\mathcal{C}}\mathcal{M}$ be a left module category over a fusion category \mathcal{C} . The internal hom is a bifunctor from $\mathcal{M} \times \mathcal{M} \rightarrow \mathcal{C}$ such that for any objects $M_1, M_2 \in \mathcal{M}$ and $X \in \mathcal{C}$, we have $\text{Hom}(X \otimes M_1, M_2) \cong \text{Hom}(X, \underline{\text{Hom}}(M_1, M_2))$. Similarly, one can define internal hom for right module categories.

The internal endomorphisms of an object M , which is defined as $\underline{\text{End}}(M) = \underline{\text{Hom}}(M, M)$, is an algebra object in \mathcal{C} .

Theorem 2.6 [Ost03]. Let M be a simple object in a semisimple module category ${}_{\mathcal{C}}\mathcal{M}$ over a fusion category \mathcal{C} . Then the category of module objects over $\underline{\text{End}}(M)$ in \mathcal{C} is equivalent to ${}_{\mathcal{C}}\mathcal{M}$ as a module category.

We can use the internal hom to give an explicit description of the inverse of a Morita equivalence ${}_{\mathcal{C}}\mathcal{K}_{\mathcal{D}}$.

Lemma 2.7 [ENO10] The inverse to a Morita equivalence ${}_{\mathcal{C}}\mathcal{K}_{\mathcal{D}}$ is the opposite category ${}_{\mathcal{D}}\mathcal{K}^{op}_{\mathcal{C}}$ (where the actions have been twisted by the duals).

Definition 2.8 If M is an object in a Morita equivalence ${}_{\mathcal{C}}\mathcal{K}_{\mathcal{D}}$ we denote the same object thought of in ${}_{\mathcal{D}}\mathcal{K}_{\mathcal{C}}^{op}$ by M^* . The reason behind this notation is that $\underline{Hom}(M, N)$ (where M and N are both in \mathcal{K}) should be thought of as $N \otimes M^*$ where M is an object in \mathcal{K}^{op} .

Given two fusion categories \mathcal{C} and \mathcal{D} , and a Morita equivalence ${}_{\mathcal{C}}\mathcal{K}_{\mathcal{D}}$ with inverse ${}_{\mathcal{D}}\mathcal{K}_{\mathcal{C}}^{op}$ one can take tensor products in several ways. For example, one can tensor an object in \mathcal{C} by an object in \mathcal{K} , or an object in \mathcal{K} by an object in \mathcal{K}^{op} . This can be formalized using the notion of a 2-category. This 2-category has two objects A and B . The 1-morphisms from A to itself are the objects of \mathcal{C} , the 1-morphisms from A to B are the objects of \mathcal{K} , the 1-morphisms from B to A are the objects of \mathcal{K}^{op} , and the 1-morphisms from B to itself are the objects of \mathcal{D} . The 2-morphisms are the 1-morphisms in \mathcal{C} , \mathcal{D} , \mathcal{K} , and \mathcal{K}^{op} .

Lemma 2.9 *The 2-category formed from a Morita equivalence ${}_{\mathcal{C}}\mathcal{K}_{\mathcal{D}}$ between two spherical fusion categories has adjoints for 1-morphisms. The adjoints of 1-morphisms from N to N or M to M are given by the dual in \mathcal{C} and \mathcal{D} , and the adjoint of a 1-morphism from N to M given by an object x in \mathcal{K} is x^* in \mathcal{K}^{op} . In particular, we have Frobenius reciprocity.*

A version of this result was proved by Müger [Müg03], for a result in much more generality (not even assuming semisimplicity) see [EGNO].

The above ideas can be made somewhat more explicit by thinking about algebra objects (which was Müger's original approach). Let A be an algebra object in a spherical fusion category \mathcal{C} and consider the Morita equivalence ${}_{\mathcal{C}}A - mod - A - mod - {}_{\mathcal{C}}$. For any left A -module X , the dual X^* has a natural right A -module structure and the map $X \mapsto X^*$ identifies ${}_{A-mod-A}mod - A_{\mathcal{C}}$ with the opposite bimodule category. There are maps $1 \rightarrow X \otimes_A X^*$ and ${}_A X^* \otimes X_A \rightarrow {}_A A_A$ satisfying the usual rigidity relation. Thus every object in an invertible bimodule category over two fusion categories has a dual object in the inverse category.

Remark 2.10 If \mathcal{K}, \mathcal{L} are categories of bifinite Hilbert bimodules over a II_1 factor, then the the inverse categories are the categories of contragredient bimodules, the relative tensor product of tensor categories is the relative tensor product of bimodules, and the existence of duals for the corresponding 2-category is Frobenius reciprocity for bimodules.

Definition 2.11 The Grothendieck ring of a fusion category \mathcal{C} is the based ring defined on the free Abelian group with basis indexed by the simple objects

of \mathcal{C} whose multiplicative structure constants on basis elements are given by the fusion rules of the category. There is a unique homomorphism from the Grothendieck ring to the real numbers which is positive on all the basis elements, called the Frobenius-Perron dimension.

We denote the Frobenius-Perron dimension of an object X in a fusion category by $\dim(X)$ or $d(X)$. If M is an object in a semisimple module category over a fusion category, we define $\dim(M) = \sqrt{\dim(\underline{\text{End}}(M))}$. If M belongs to a bimodule category over two fusion categories, the left and right internal endomorphisms have the same dimensions so there is an unambiguous dimension associated to M .

2.2 Subfactors

A (II_1) subfactor is a unital inclusion $N \subset M$ of II_1 factors. The subfactor is said to have *finite index* if M is a finitely generated projective module over N [Jon83, PP86]. In this case, letting $\kappa = {}_N M_M$ and $\bar{\kappa} = {}_M M_N$, the bimodules $\kappa \otimes_M \bar{\kappa}$ and $\bar{\kappa} \otimes_N \kappa$ generate C^* -tensor categories of $M - M$ and $N - N$ bimodules, respectively; these tensor categories are called the *principal and dual even parts* of the subfactor. The subfactor is said to have *finite depth* if each of the even parts has only finitely many simple objects up to isomorphism - in this case the even parts are unitary fusion categories, and the category of $N - M$ bimodules generated by the even parts and κ is a Morita equivalence between them. We will often use sector notation, in which object in categories are represented by lowercase Greek letters, tensor product symbols are suppressed and $(\kappa, \lambda) := \dim(\text{Hom}(\kappa, \lambda))$.

Definition 2.12 [Lon94, LR97]. A Q -system in a C^* -tensor category is an algebra object such that the multiplication map is a coisometry. A Q -system has a dimension which coincides with the Frobenius-Perron dimension of the algebra object in the case of a unitary fusion category.

For a finite index subfactor as above, the bimodules $\kappa \otimes_M \bar{\kappa}$ and $\bar{\kappa} \otimes_N \kappa$ have natural Q -system structures. Moreover, any Q -system in a C^* tensor category can be realized in this way from a subfactor [Lon94, LR97, Yam03]. The index of the subfactor is the dimension of the Q -system $\kappa \otimes_M \bar{\kappa}$.

Definition 2.13 Let $N \subset M$ be a subfactor with fundamental bimodule $\kappa = {}_N M_M$, principal and dual even parts \mathcal{C} and \mathcal{D} , and Morita equivalence ${}_{\mathcal{C}}\mathcal{H}_{\mathcal{D}}$

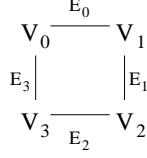


Figure 1: Schematic representation of a 4-graph.

generated by $\mathcal{C}\kappa\mathcal{D}$. The principal graph of the subfactor is the bipartite graph with even vertices indexed by \mathcal{C} , odd vertices indexed by \mathcal{K} , and $(\xi\kappa, \lambda)$ edges connecting each even vertex ξ with each odd vertex λ . The dual graph is defined similarly but using the dual even part \mathcal{D} (which acts on the right of κ) instead.

2.3 Connections

The theory of connections on graphs was introduced by Ocneanu [Ocn88]. A good resource is [EK98]. See also [AH99].

Definition 2.14 By a 4-graph we mean a graph with 4 finite sets of vertices V_i , for $i \in \mathbb{Z}/4\mathbb{Z}$ and 4 finite sets of edges E_i , such that each edge e in E_i connects a vertex in V_i (called the source $s(e)$) with a vertex in $V(i+1)$ (called the target, $t(e)$). A cell in a 4 graph is a choice of edges $e_i \in E_i$ such that $s(e_{i+1}) = t(e_i)$. A connection on a 4-graph is an assignment of a complex number to each cell.

We think of the edges as being placed in a square, clockwise with 0 at the top left. We then call $V_0 - E_0 - V_1$ the top graph, $V_1 - E_1 - V_2$ the right graph, etc.

Given a finite depth subfactor $N \subset M$, one can define a flat, binunitary connection on a 4-graph whose upper and left graphs are each the principal graph of the subfactor and whose lower and right graphs are each the dual graph of the subfactor. Moreover, any flat, binunitary connection on a 4-graph arises this way.

Definition 2.15 The edge space H_E^G of a 4-graph G is the Hilbert space with orthonormal basis indexed by the edges of the 4-graph. For every pair of vertices $v \in V_i, w \in V_{i+1}$, we have a subspace $H_{v,w}^G \subset H_E^G$ spanned by the edges $\{e | s(e) = v, t(e) = w\}$. An edge space map between two 4-graphs G_1, G_2 with the same vertex sets $\{V_i\}$ is a linear map $T : H_E^{G_1} \rightarrow H_E^{G_2}$ such

that $T(H_{v,w}^{G_1}) \subseteq H_{v,w}^{G_2}, \forall v, w$. A gauge transformation on a 4-graph G is an edge space map from G to itself. A vertical gauge transformation is a gauge transformation which fixes all of the horizontal edge spaces.

A connection on a 4-graph can be extended (linearly for the left and lower graphs, antilinearly for upper and right graphs) to cells consisting of vectors in the edge space. Then composing the connection with any gauge transformation gives a new connection.

In [AH99] it was shown that any binitary connection on a 4-graph defines a bimodule over the II_1 factors generated by the string algebras of the upper and lower graphs. Moreover, sums and compositions of connections were defined which correspond to direct sums and tensor product of bimodules.

Theorem 2.16 [AH99] *Two connections on 4-graphs with the same connected horizontal graphs give isomorphic bimodules iff the vertical graphs are also the same and the connections are equivalent up to a unitary vertical gauge transformation.*

This gives a concrete realization of the unitary fusion categories coming from a subfactor: one starts with the flat connection on the principal and dual graphs, which corresponds to the fundamental bimodule $\kappa = {}_N M_M$. Then by decomposing powers of κ and $\bar{\kappa}$ into direct sums of irreducible connections, one obtains the connections corresponding to the simple objects in the even parts of the subfactor (the “even” bimodules), as well as those corresponding to the simple objects in the Morita equivalence (the “odd” bimodules). Finally, morphisms in these categories, which are intertwiners of the bimodules, can be represented as edge space maps of the corresponding connection 4-graphs.

3 Diagrams for algebra objects

In this section we prove two lemmas which we will need later on. Both use diagrammatic techniques. The first lemma shows that certain objects automatically admit algebra structures, and the second allows us to characterize when $\kappa\bar{\kappa}$ and $\lambda\bar{\lambda}$ are isomorphic as algebra objects.

3.1 Intertwiner diagrams

Following [Pen71, RT91, JS91, Jon], we will often use diagrams for computations in tensor categories. Morphisms are represented by vertices or boxes,

from which emanate strings labeled by the source objects (upwards) and by the target objects (downwards). Straight strings labeled by objects correspond to identity morphisms, and strings labeled by identity objects are often suppressed. Tensoring is depicted by horizontal concatenation, and composition by vertical concatenation. Diagrams are read from top to bottom. Then various planar isotopies can be applied to the diagram according to the duality rules of the category.

Let ξ be an object in a fusion category with left and right duals ${}^*\xi$ and ξ^* , respectively. Then by the rigidity property, there are morphism $\eta_\xi^r : 1 \rightarrow \xi \otimes \xi^*$, $\epsilon_\xi^r : 1 \rightarrow \xi^* \otimes \xi$ such that $(Id_\xi \otimes \epsilon_\xi^r) \circ (\eta_\xi^r \otimes Id_\xi) = Id_\xi$. (Here we suppress the associativity and unit isomorphisms.) Similarly, there are morphisms $\eta_\xi^l : 1 \rightarrow {}^*\xi \otimes \xi$, $\epsilon_\xi^l : 1 \rightarrow \xi \otimes {}^*\xi$ such that $(\epsilon_\xi^l \otimes Id_\xi) \circ (Id_\xi \otimes \eta_\xi^l) = Id_\xi$.

Diagrammatically, we write:

$$\begin{array}{c} \text{arc from } \xi^* \text{ to } \xi \\ \hline \end{array} = \eta_\xi^l, \quad \begin{array}{c} \text{arc from } \xi \text{ to } {}^*\xi \\ \hline \end{array} = \eta_\xi^r, \quad \begin{array}{c} \text{arc from } \xi \text{ to } \xi^* \\ \hline \end{array} = \epsilon_\xi^l, \quad \begin{array}{c} \text{arc from } {}^*\xi \text{ to } \xi \\ \hline \end{array} = \epsilon_\xi^r$$

Then the duality relations are expressed as:

$$\begin{array}{c} \xi \\ | \\ \text{arc from } \xi \text{ to } {}^*\xi \\ | \\ \xi \end{array} = \begin{array}{c} \xi \\ | \\ \xi \end{array}, \quad \begin{array}{c} \xi \\ | \\ \text{arc from } {}^*\xi \text{ to } \xi \\ | \\ \xi \end{array} = \begin{array}{c} \xi \\ | \\ \xi \end{array}$$

If the fusion category is unitary, then ${}^*\xi \cong \xi^*$ and we write $\bar{\xi}$ for the dual object and reserve the $*$ symbol for the unitary conjugation on morphisms. Let $\xi = \bar{\xi}$ be a self-dual object in a unitary fusion category. We may choose $\epsilon_\xi^r = (\eta_\xi^l)^*$ and $\epsilon_\xi^l = (\eta_\xi^r)^*$, and then there is a number $c_\xi \in \{\pm 1\}$ such that $\eta_\xi^l = c_\xi \eta_\xi^r$. If $c = 1$ we say that ξ is symmetrically self-dual, and we write

$$\begin{array}{c} \text{arc from } \xi \text{ to } \xi \\ \hline \end{array}, \quad \begin{array}{c} \text{arc from } \xi \text{ to } \xi \\ \hline \end{array}$$

for the common left and right duality maps. The invariant c is multiplicative in the sense that if ξ, μ, ν are self-dual objects in a unitary fusion category such that $(\xi, \mu\nu) \neq 0$, then $c_\xi = c_\mu c_\nu$.

Finally we recall the following computation for Q-systems corresponding to 2-supertransitive algebra objects.

Lemma 3.1 [GI08] Let σ be a symmetrically self-dual simple object in a unitary fusion category with $d = \dim(\sigma) \neq 1$. Fix a duality map $\sigma \curvearrowright \sigma$. Then

$1 + \sigma$ admits a Q -system if there is an isometry $d^{-\frac{1}{4}}$ $\begin{array}{c} \sigma \\ \diagup \quad \diagdown \\ \sigma \quad \sigma \end{array}$ such that the following relations are satisfied:

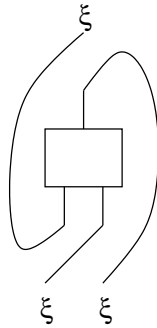
$$(1) \quad \begin{array}{c} \sigma \\ \diagup \quad \diagdown \\ \sigma \quad \sigma \end{array} \sigma = \sigma \begin{array}{c} \sigma \\ \diagup \quad \diagdown \\ \sigma \quad \sigma \end{array}$$

$$(2) \quad \frac{\sqrt{d+1}}{d} \left(\begin{array}{c} \sigma \\ \diagup \quad \diagdown \\ \sigma \quad \sigma \end{array} \sigma - \sigma \begin{array}{c} \sigma \\ \diagup \quad \diagdown \\ \sigma \quad \sigma \end{array} \right) = \begin{array}{c} \sigma \\ \diagup \quad \diagdown \\ \sigma \quad \sigma \end{array} \sigma - \sigma \begin{array}{c} \sigma \\ \diagup \quad \diagdown \\ \sigma \quad \sigma \end{array}.$$

3.2 4-supertransitive objects admit Q -systems

In this section we prove that if $\xi^2 \cong 1 + \xi + \mu$, where μ is a simple object with $d(\mu) > 1$, then $1 + \xi$ admits an algebra structure. This is a Wenzl-style recognition theorem and our argument is inspired by similar results in [KW93, TW05, MPS].

Let ξ be a symmetrically self-dual object in a unitary fusion category. Then the rotation operator



acts as a period 3 automorphism on the vector space $\text{Hom}(\xi, \xi^2)$. If $(\xi, \xi^2) = 1$ then the automorphism is necessarily scalar multiplication by a cube root of unity; this cube root of unity is called the rotational eigenvalue of ξ .

Lemma 3.2 *Let ξ be an object in a unitary fusion category such that $\xi^2 \cong 1 + \xi + \mu$, where μ is a simple object with $d = d(\mu) > 1$. Then ξ is symmetrically self-dual with a rotational eigenvalue of 1.*

Proof Since $(\xi, \xi^2) = 1$, we have $c_\xi = c_\xi^2$, so $c_\xi = 1$ and ξ is symmetrically self-dual. Fix a duality map η_ξ and an isometry

$$\begin{array}{c} \xi \\ \diagdown \quad \diagup \\ \xi \quad \xi \end{array} = v \in \text{Hom}(\xi, \xi^2)$$

Let

$$\begin{array}{c} \xi \quad \xi \\ \diagdown \quad \diagup \\ \xi \end{array} = v^*.$$

Then the following projections form an orthogonal basis of $\text{End}(\xi^2)$:

$$e_1 = \frac{1}{d} \begin{array}{c} \xi \quad \xi \\ \text{---} \\ \xi \quad \xi \end{array}, \quad e_2 = \begin{array}{c} \xi \quad \xi \\ \diagdown \quad \diagup \\ \xi \quad \xi \end{array}, \quad \text{and} \quad e_3 = 1 - e_1 - e_2.$$

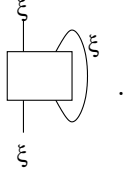
Assume for the sake of contradiction that the rotational eigenvalue is not 1. Consider the diagram

$$\begin{array}{c} \xi \\ \diagdown \quad \diagup \\ \xi \quad \xi \end{array} \cdot \begin{array}{c} \xi \quad \xi \\ \diagdown \quad \diagup \\ \xi \quad \xi \end{array}.$$

Note that this diagram is rotationally invariant (since any rotation picks up a cube of the rotational eigenvalue). Therefore, since $(\xi^2, \xi) = 1$, we see that it must be zero. Let

$$x = \begin{array}{c} \xi \quad \xi \\ \diagdown \quad \diagup \\ \xi \quad \xi \end{array} := \begin{array}{c} \xi \quad \xi \\ \diagdown \quad \diagup \\ \xi \quad \xi \end{array} \in \text{End}(\xi^2).$$

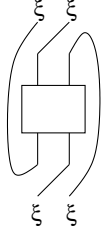
As we saw a moment ago $xe_2 = 0$. Let $tr : End(\xi^2) \rightarrow \mathbb{C}$ be the coefficient of the identity under the action of the linear operator



Then $tr(e_1) = \frac{1}{d}$, $tr(e_2) = 1$, $tr(e_3) = d - 1 - \frac{1}{d}$. Also we have $xe_1 = e_1$ and $tr(x) = 0$. Write $x = e_1 + be_3$. Then we have $0 = tr(x) = \frac{1}{d} + b(d - 1 - \frac{1}{d})$. Solving for b and gathering terms, we get the linear relation

$$\begin{array}{c} \xi \quad \xi \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ \xi \quad \xi \end{array} - \frac{1}{d^2 - d - 1} \begin{array}{c} \xi \quad \xi \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ \xi \quad \xi \end{array} = \frac{d-1}{d^2 - d - 1} \begin{array}{c} \xi \quad \xi \\ \text{---} \\ \xi \quad \xi \end{array} - \frac{1}{d^2 - d - 1} \begin{array}{c} \xi \quad \xi \\ \text{---} \\ \xi \quad \xi \end{array}.$$

The rotation operator



acts on this linear relation by permuting the

two diagrams on the left and permuting the two diagrams on the right, and multiplying the rotation by $1 + d - d^2$ gives

$$a \begin{array}{c} \xi \quad \xi \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ \xi \quad \xi \end{array} - (d^2 - d - 1) \begin{array}{c} \xi \quad \xi \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ \xi \quad \xi \end{array} = \begin{array}{c} \xi \quad \xi \\ \text{---} \\ \xi \quad \xi \end{array} - (d-1) \begin{array}{c} \xi \quad \xi \\ \text{---} \\ \xi \quad \xi \end{array}.$$

By the linear independence of e_1, e_2, e_3 this is only possible if $d = 1$. This gives a contradiction; thus the rotational eigenvalue must be 1. \square

Remark 3.3 Note that the assumption that $d(\mu) > 1$ is necessary - if $d(\mu) = 1$ there are three fusion categories with those fusion rules coming from twisting Rep_{S_3} in different ways, and the three fusion categories yield three different rotational eigenvalues.

Theorem 3.4 *Let ξ be an object in a unitary fusion category such that $\xi^2 \cong 1 + \xi + \mu$, where μ is a simple object with $d(\mu) > 1$. Then $1 + \xi$ admits a Q-system.*

Proof Using the same notation as in the previous proof, we write

$x = e_1 + ae_2 + be_3$ and use the rotation and linear independence to solve for $b = \frac{1}{1+d}$, $a = b \pm 1$. Taking traces of the equation determines the sign, and we get

$$\begin{array}{c} \xi & \xi \\ & \diagdown \quad \diagup \\ & \text{---} \\ & \diagup \quad \diagdown \\ \xi & \xi \end{array} - \begin{array}{c} \xi & \xi \\ & \diagdown \quad \diagup \\ & \text{---} \\ & \diagup \quad \diagdown \\ \xi & \xi \end{array} = \frac{1}{d+1} \left(\begin{array}{c} \xi & \xi \\ & \text{---} \\ \xi & \xi \end{array} + \begin{array}{c} \xi & \xi \\ & \text{---} \\ \xi & \xi \end{array} \right).$$

Using this and the rotational invariance of $\begin{array}{c} \xi \\ & \diagdown \quad \diagup \\ & \text{---} \\ & \diagup \quad \diagdown \\ \xi & \xi \end{array}$ the Q-system relations

for $1 + \xi$ can be easily verified using Lemma 3.1. \square

3.3 Algebra isomorphisms come from invertible objects in the dual category

Let κ be an object in an invertible bimodule category ${}_{\mathcal{C}}\mathcal{K}_{\mathcal{D}}$ over two fusion categories. Then there is an object $\bar{\kappa}$ in the inverse category ${}_{\mathcal{D}}\mathcal{K}_{\mathcal{C}}$ such that there are unit maps $\eta_{\kappa} : 1 \rightarrow \kappa \otimes_{\mathcal{D}} \bar{\kappa}$ and $\eta_{\bar{\kappa}} : 1 \rightarrow \bar{\kappa} \otimes_{\mathcal{C}} \kappa$ satisfying the usual duality relations, which together with the multiplication maps

$$\begin{array}{c} \kappa & \bar{\kappa} & \kappa & \bar{\kappa} \\ & \diagdown \quad \diagup \\ & \text{---} \\ & \diagup \quad \diagdown \\ \kappa & \bar{\kappa} \end{array}, \quad \begin{array}{c} \bar{\kappa} & \kappa & \bar{\kappa} & \kappa \\ & \diagdown \quad \diagup \\ & \text{---} \\ & \diagup \quad \diagdown \\ \bar{\kappa} & \kappa \end{array}$$

form the algebras of internal endomorphisms of κ in \mathcal{C}, \mathcal{D} , respectively.

Definition 3.5 An algebra isomorphism between two algebra objects ξ, η in a fusion category is an isomorphism between ξ and η which commutes with the multiplication and unit maps on ξ, η .

If we express the algebra objects as $\xi = \kappa\bar{\kappa}, \eta = \lambda\bar{\lambda}$ for two module objects and write the isomorphism as

$$\begin{array}{c} \kappa \quad \bar{\kappa} \\ | \quad | \\ \boxed{X} \\ | \quad | \\ \lambda \quad \bar{\lambda} \end{array},$$

the algebra isomorphism conditions become

$$\begin{array}{c} \kappa \\ | \\ \boxed{X} \\ | \quad | \\ \lambda \quad \bar{\lambda} \end{array} = \begin{array}{c} \text{cap} \\ | \\ \lambda \quad \bar{\lambda} \end{array} \quad \text{and} \quad \begin{array}{c} \kappa \quad \bar{\kappa} \quad \kappa \quad \bar{\kappa} \\ | \quad | \quad | \quad | \\ \boxed{X} \quad \boxed{X} \\ | \quad | \quad | \quad | \\ \lambda \quad \bar{\lambda} \quad \lambda \quad \bar{\lambda} \end{array} = \begin{array}{c} \bar{\kappa} \quad \bar{\kappa} \quad \kappa \quad \kappa \\ | \quad | \quad | \quad | \\ \text{cup} \\ | \\ \boxed{X} \\ | \quad | \\ \lambda \quad \bar{\lambda} \end{array}.$$

Theorem 3.6 *Let $\xi = \kappa\bar{\kappa}$ and $\eta = \lambda\bar{\lambda}$ be isomorphic algebra objects in a fusion category \mathcal{C} , where κ and λ are objects in a left module category ${}_{\mathcal{C}}\mathcal{K}$. Then there is an invertible object α in the dual category \mathcal{D} such that $\kappa\alpha \cong \lambda$.*

Proof Let

$$\begin{array}{c} \bar{\kappa} \quad \lambda \\ | \quad | \\ \boxed{X} \\ | \quad | \\ \bar{\kappa} \quad \lambda \end{array}$$

be the image of the algebra isomorphism under rotation, and let $d = \dim(\kappa) =$

$dim(\lambda)$. Then from the algebra isomorphism multiplication relation we obtain:

$$\begin{array}{c} \bar{\kappa} \quad \lambda \\ \downarrow \\ \boxed{x} \\ \downarrow \\ \boxed{x} \\ \downarrow \\ \bar{\kappa} \quad \lambda \end{array} = d \begin{array}{c} \bar{\kappa} \quad \lambda \\ \downarrow \\ \boxed{x} \\ \downarrow \\ \bar{\kappa} \quad \lambda \end{array} .$$

Thus we see that

$$\frac{1}{d} \begin{array}{c} \bar{\kappa} \quad \lambda \\ \downarrow \\ \boxed{x} \\ \downarrow \\ \bar{\kappa} \quad \lambda \end{array}$$

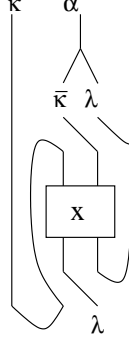
is an idempotent. Call the object which is the image of the idempotent α . From the algebra isomorphism unit relation we get:

$$dim(\alpha) = \frac{1}{d} \begin{array}{c} \bar{\kappa} \quad \lambda \\ \downarrow \\ \boxed{x} \\ \downarrow \\ \bar{\kappa} \quad \lambda \end{array} = 1.$$

So α is invertible. Let

$$\alpha$$

be an inclusion map. Then



is the desired isomorphism from $\kappa\alpha$ to λ .

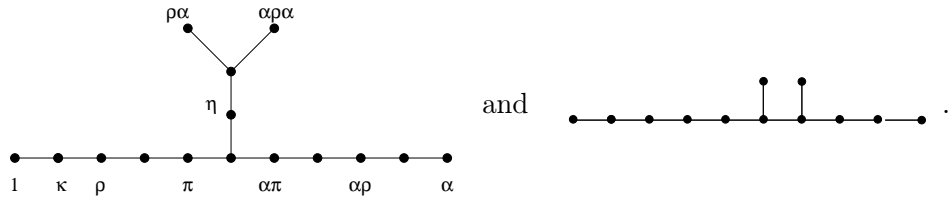
□

4 The Asaeda-Haagerup categories

In this section we recall some results about the Asaeda-Haagerup subfactor [AH99], and the related AH+1 subfactor [AG11]. In addition we sketch the construction of a new subfactor, the AH+2 subfactor, using the techniques from those two papers.

4.1 AH and AH+1

The Asaeda-Haagerup subfactor, constructed in [AH99], is a finite depth subfactor with index $\frac{5+\sqrt{17}}{2}$ and principal graphs

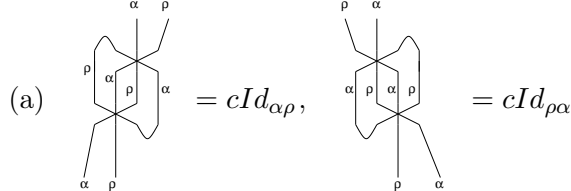


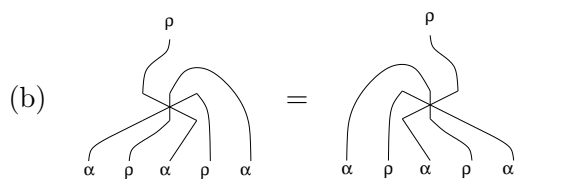
Here we label the simple objects in the principal even part as well as the generating module object κ . As part of their construction of the subfactor, Asaeda and Haagerup explicitly wrote down the unique (up to gauge choice) connection for κ , the connection ρ defined (up to vertical gauge choice) by $1 + \rho \cong \kappa\bar{\kappa}$, and the connection for the automorphism α (which is again uniquely determined

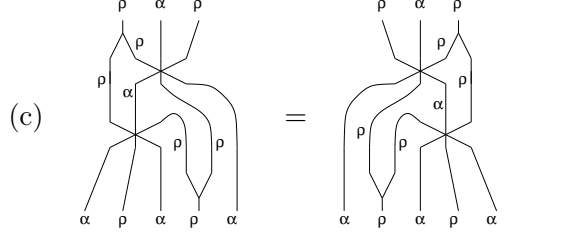
up to vertical gauge choice, and can be taken to be identically 1.) Then they wrote down a vertical gauge transformation (i.e. an intertwiner) between $\rho\alpha\kappa$ and $\alpha\rho\alpha\kappa$.

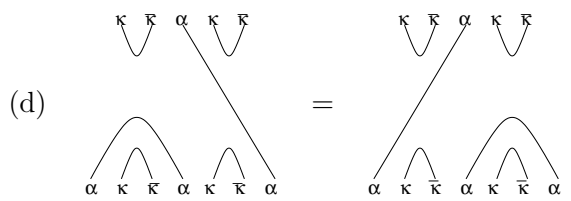
Motivated by the study of quadrilaterals of factors, Izumi conjectured the existence of a Q-system for the object $1 + \bar{\kappa}\alpha\kappa$ in the dual even part. This conjecture was verified in [AG11] by showing that the following diagrammatic relations hold, and that they lead a solution to the Q-system equations 3.1 for $1 + \bar{\kappa}\alpha\kappa$:

4.1 The Asaeda-Haagerup algebra relations:

(a) 

(b) 

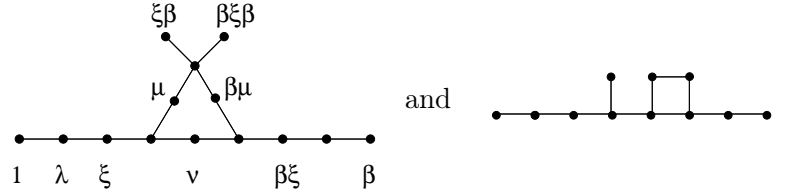
(c) 

(d) 

Note that although the vertices of the intertwiner diagrams are each only determined up to a scalar, the relations make sense independently of a choice of scalars, since the same vertices appear on both sides of each equation (except for equations (a), which contain an arbitrary scalar on the right hand side.)

To establish these relations, it was necessary to evaluate the diagrams on various “states”, i.e. labelings of the diagrams by vertices and edges from the connection 4-graphs of the appropriate objects. The states were evaluated by decomposing the diagrams into tensor products and compositions of the elementary intertwiners $1 \rightarrow \alpha^2$, $1 \rightarrow \kappa\bar{\kappa}$, $1 \rightarrow \bar{\kappa}\kappa$, $\rho \rightarrow \kappa\bar{\kappa}$, and $\rho\alpha\kappa \rightarrow \alpha\rho\alpha\kappa$. These elementary intertwiners act on edges by explicit formulas given by gauge transformation matrices.

As a consequence of the existence of the Q-system, we obtain a subfactor with index $\frac{7+\sqrt{17}}{2}$ and graphs



Again we label the simple objects in the principal even part as well as the generating module object λ .

4.2 Existence of AH+2

Because of the similar fusion structure of this new subfactor, it was conjectured in [AG11] that the procedure could be iterated once more to obtain a Q-system for the object $1 + \bar{\lambda}\beta\lambda$ in the dual even part (which is the same category as the dual even part of the original Asaeda-Haagerup subfactor.)

It turns out that this is indeed the case, and we briefly sketch the argument here. The computation consists of two parts: first we must replicate Asaeda and Haagerup’s computation of the connections for λ, ξ, β and the intertwiner $\xi\beta\lambda \rightarrow \beta\xi\beta\lambda$. Then we must verify that the corresponding relations 4.1 above hold for these connections.

In Figure 4.2 we indicate what the 4 graphs are, using a labeling and display similar to that used by [AH99]. Note that in the figure we have “unwrapped the square”, so reading from top to bottom, we have first the upper, then the right, then the lower, then the left graphs.

Note that if $N \subset M$ is the subfactor with Q-system $\lambda\bar{\lambda}$, then the 4-graphs for the connections of any of the associated $N - M$ (resp. $N - N$) bimodules has the same horizontal graphs (the first and third levels form the top in Figure 4.2)

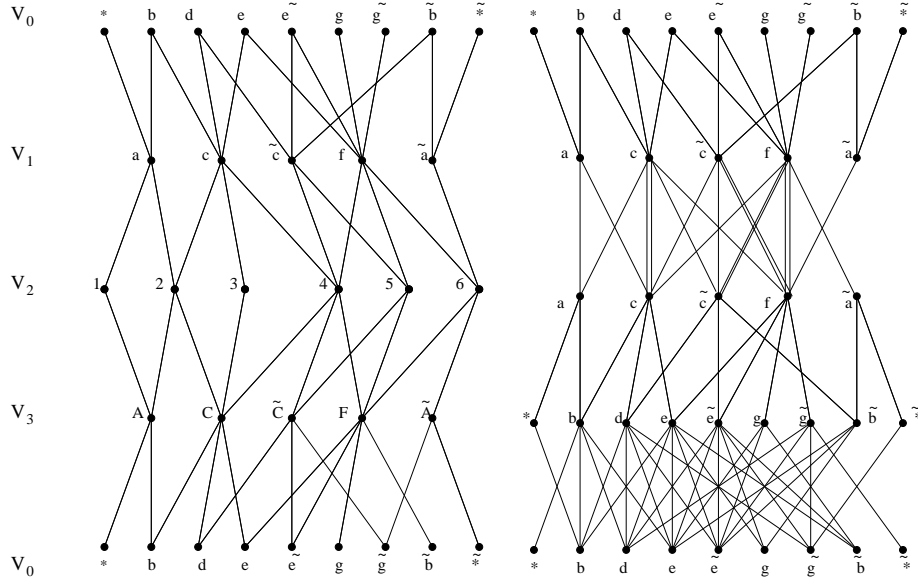


Figure 2: The 4-graphs for the connections of λ (left) and ξ (right)

as that of λ (resp. ξ .) Therefore to specify the 4-graph of such a bimodule it suffices to specify the vertical graphs (the second and fourth levels in Figure 4.2.)

The vertical graphs of the 4-graph for the identity $N - N$ bimodule have exactly one edge emanating from each vertex in V_1, V_3 connecting to the vertex with the same labeling in V_{i+1} . The vertical graphs of the 4-graph for the $N - N$ bimodule β have exactly one edge emanating from each vertex x in V_1, V_3 connecting to the vertex $\tilde{x} \in V_{i+1}$, where $\tilde{x} = x$ and $\tilde{x} = x$ if there is no vertex labeled \tilde{x} .

Lemma 4.2 (a) *There is a unique connection on the 4-graph for λ up to gauge choice, which may be taken to be real.*

(b) *There are exactly two real connections on the 4-graph for the automorphism β up to vertical gauge choice - one is identically 1, the other has the value -1 on one cell.*

(c) *The connection for β is not identically 1.*

The uniqueness of the connection for λ was first pointed out to the authors by Marta Asaeda, who also corrected some signs in the computation.

We now describe a version of the connection for λ using the following notation. We refer to 4.2 for the labelings of vertices. Then the connection is given by matrices corresponding to pairs $u - v$ with $u \in V_0$ and $v \in V_2$, where the rows and columns are indexed by V_3 and V_1 , respectively. (You should read $u - v$ as “between u and v ” not “ u minus v .”) These matrices are necessarily square whenever they are nonempty.

In this case the connection consists of 5 2×2 matrices and a bunch of 1×1 matrices; for the 1×1 matrices we suppress the matrix notation and simply refer to the entry as $u - v$. Following [AH99] we introduce the positive numbers

$\beta_n = \sqrt{\frac{7 + \sqrt{17}}{2}} - n$, $n \leq 5$. Then the connection is:

$$\begin{array}{c|cc} b-2 & a & c \\ \hline A & \frac{-1}{\beta_1^2} & \frac{\beta\beta_2}{\beta_1^2} \\ C & \frac{\beta\beta_2}{\beta_1^2} & \frac{1}{\beta_1^2} \end{array}
\quad
\begin{array}{c|cc} d-4 & c & \tilde{c} \\ \hline C & \frac{-1}{\beta_{-1}} & \frac{\tilde{c}}{\beta_{-1}} \\ \tilde{C} & \frac{-\beta}{\beta_{-1}} & \frac{-1}{\beta_{-1}} \end{array}
\quad
\begin{array}{c|cc} e-4 & c & f \\ \hline C & \frac{-\beta_5}{\beta_1\beta_3} & \frac{\sqrt{2}\beta}{\beta_1\beta_3} \\ F & \frac{\sqrt{2}\beta}{\beta_1\beta_3} & \frac{\beta_5}{\beta_1\beta_3} \end{array}$$

$$\begin{array}{c|cc} \tilde{e}-4 & \tilde{c} & f \\ \hline \tilde{C} & \frac{2}{\beta_{-1}} & \frac{-\beta_3}{\beta_{-1}} \\ F & \frac{\beta_3}{\beta_{-1}} & \frac{2}{\beta_{-1}} \end{array}
\quad
\begin{array}{c|cc} \tilde{e}-5 & \tilde{c} & f \\ \hline \tilde{C} & \frac{2}{\beta_1^2} & \frac{\sqrt{2}\beta_{-1}}{\beta_1\beta_2} \\ F & \frac{-\sqrt{2}\beta_{-1}}{\beta_1\beta_2} & \frac{2}{\beta_1^2} \end{array}$$

The 1×1 entries $e - 2$, $\tilde{e} - 6$, and $g - 5$ are -1 ; all the other 1×1 entries are 1.

For β all the matrices are 1×1 . We take all the entries to be 1 except for $e - f$, which we take to be -1 . That β cannot be identically 1 (part (c) of the lemma above) was discovered by trial and error. Then we compute a version of the connection for ξ , which is uniquely determined up to vertical gauge choice.

With this information, we can compute all the necessary elementary intertwiners:

Lemma 4.3 *Intertwiners $1 \rightarrow \beta^2, 1 \rightarrow \lambda\bar{\lambda}, 1 \rightarrow \bar{\lambda}\lambda, \xi \rightarrow \lambda\bar{\lambda}$, and $\xi\beta\lambda \rightarrow \beta\xi\beta\lambda$ can be computed.*

Proof These are just (lengthy) computations. The similar calculation in [AH99] takes almost 30 pages, and this calculation is somewhat more involved. We used Mathematica for bookkeeping and to multiply matrices of algebraic numbers. \square

One notable difference from the original Asaeda-Haagerup case is that since the connection for the automorphism is not identically 1 here, the intertwiner $1 \rightarrow \beta^2$ is nontrivial.

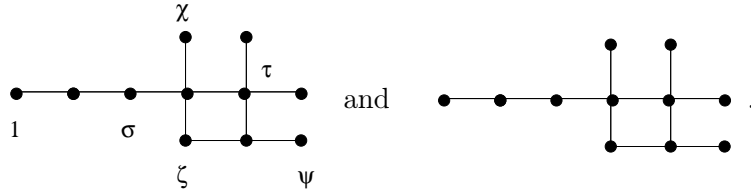
Lemma 4.4 *The relations 4.1 hold when ρ, α, κ are replaced by ξ, β, λ respectively.*

Proof Again, by direct computation, following the methods of [AG11]. \square

Theorem 4.5 *The object $1 + \bar{\lambda}\beta\lambda$ admits a Q -system.*

Proof As in [AG11] we may choose c in relation (a) to be $\sqrt{d(\xi)}$, and the proof proceeds exactly as in the main theorem there. \square

By the above result we now have a third subfactor with index $\frac{9+\sqrt{17}}{2}$ and graphs



4.3 Relations among the Asaeda-Haagerup fusion categories

The three subfactors discussed in the preceeding sections all have the same dual even part but different principal even parts. That the principal even parts are different can be seen immediately by checking the Frobenius-Perron weights of the even vertices of the principal graphs, which are different for the three different graphs. In fact, the complete fusion rules for each of the three principal even parts may be deduced from the corresponding graph. The dual data is all trivial except that $\rho\alpha$ is dual to $\alpha\rho$ and similarly $\xi\beta$ is dual to $\beta\xi$. We include the multiplication tables below.

We will call the three subfactors AH, AH+1, and AH+2. We will call the fusion category which is the dual even part of all three subfactors \mathcal{AH} , the principal even part of AH+2 \mathcal{AH}_1 , the principal even part of AH \mathcal{AH}_2 , and the principal even part of AH+1 \mathcal{AH}_3 . The Grothedieck rings of these fusion categories will be called $AH = AH1$, $AH2$, and $AH3$ respectively.

	ψ	χ	σ	ζ	τ
ψ	$1 + \psi + \zeta$	$\chi + \tau$	$\zeta + \tau$	$\psi + \sigma + \zeta + \tau$	$\chi + \sigma + \zeta + 2\tau$
χ		$1 + \psi + \chi + \tau$	$\sigma + \zeta + \tau$	$\sigma + \zeta + 2\tau$	$\Lambda + \zeta + \tau$
σ			$1 + \chi + \sigma + \zeta + \tau$	$\Lambda + \tau$	$\Lambda + \zeta + 2\tau$
ζ				$1 + \Lambda + \zeta + 2\tau$	$\Lambda + \chi + \sigma + 2\zeta + 3\tau$
τ					$1 + 2\Lambda + \sigma + 2\zeta + 4\tau$

We use the abbreviation $\Lambda = \psi + \chi + \sigma + \zeta + \tau$. Since the multiplication is commutative, we omit the sub-diagonal entries.

Table 1: *AH1* multiplication table.

	ρ	π	η
ρ	$1 + \rho + \pi$	$\rho + \Gamma$	$\alpha\rho + \alpha\rho\alpha + \Gamma$
π	$\rho + \Gamma$	$1 + \Delta + 2\Gamma$	$\Delta + 2\Gamma + \eta$
η	$\rho\alpha + \alpha\rho\alpha + \Gamma$	$\Delta + 2\Gamma + \eta$	$1 + \alpha + \Delta + 2\Gamma + \pi + \alpha\pi$

Rules involving α : $\alpha^2 = 1$, $\pi\alpha = \alpha\pi$, $\alpha\eta = \eta\alpha = \eta$, $\rho\alpha\rho = \alpha\rho\alpha + \eta$.

We use the abbreviations $\Gamma = \pi + \alpha\pi + \eta$, $\Delta = \rho + \alpha\rho + \rho\alpha + \alpha\rho\alpha$.

Table 2: *AH2* partial multiplication table.

	ξ	μ	ν
ξ	$1 + \xi + \mu + \nu$	$\xi + \beta\xi + \beta\xi\beta + \Pi$	$\xi + \xi\beta + \Pi$
μ	$\xi + \xi\beta + \beta\xi\beta + \Pi$	$1 + \Pi + 2\Psi$	$\Pi + \Psi + \mu + \beta\mu$
ν	$\xi + \beta\xi + \Pi$	$\Pi + \Psi + \mu + \beta\mu$	$1 + \beta + \Pi + \Psi + \nu$

Rules involving β : $\beta^2 = 1$, $\mu\beta = \beta\mu$, $\beta\nu = \nu\beta = \nu$, $\xi\beta\xi = \beta\xi\beta + \mu + \beta\mu$.

We use the abbreviation $\Pi = \mu + \beta\mu + \nu$, $\Psi = \xi + \beta\xi + \xi\beta + \beta\xi\beta$.

Table 3: *AH3* partial multiplication table.

Lemma 4.6 *Let \mathcal{C} be a unitary fusion category containing an object ξ such that the Frobenius-Perron dimension $\dim(\xi) = \frac{3+\sqrt{17}}{2}$ and such that $\xi^2 \cong 1 + \xi + \eta$ where η is a simple object. Then \mathcal{C} is equivalent to either \mathcal{AH}_1 or \mathcal{AH}_2 .*

Proof This follows immediately from Theorem 3.4 and the uniqueness (up to duality) of the finite-depth subfactor with index $\frac{5+\sqrt{17}}{2}$. \square

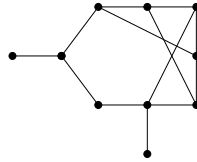
Corollary 4.7 *We have $\mathcal{AH} \cong \mathcal{AH}_1$. Thus, $AH+2$ is an autoequivalence.*

Proof The object ψ and the corresponding object on the dual graph both satisfy the conditions of the previous lemma, and clearly neither even part is \mathcal{AH}_2 . \square

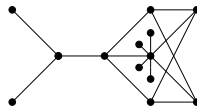
Note that tensoring with α fixes the middle vertex of the $AH2$ graph, and hence the tensor subcategory generated by α has Vec as a module category and thus has trivial associator. As a consequence, $1 + \alpha$ has a unique algebra structure given by the group ring of $\mathbb{Z}/2\mathbb{Z}$.

Theorem 4.8 *The category of bimodules over the two-dimensional algebra object $1 + \alpha$ in \mathcal{AH}_2 is equivalent to \mathcal{AH}_3 . Similarly, \mathcal{AH}_2 is equivalent to the category of $(1 + \beta) - (1 + \beta)$ bimodules in \mathcal{AH}_3 .*

Proof Let λ be an $\mathcal{AH}_3 - \mathcal{AH}_1$ bimodule corresponding to the Asaeda-Haagerup subfactor, as above. Let μ be an object in an invertible $\mathcal{C} - \mathcal{AH}_3$ bimodule category such that $\bar{\mu}\mu = 1 + \beta$. Then since $(1 + \beta, 1 + \xi) = 1$, by Frobenius reciprocity $\mu\lambda$ is an irreducible $\mathcal{C} - \mathcal{AH}_3$ bimodule. We can compute (see the discussion of fusion computations in Section 5) that the dual graph of $\mu\lambda$ must be



and then that the principal graph must be



This implies that the simple objects of \mathcal{C} have the same Frobenius-Perron dimension as those of \mathcal{AH}_2 . We can compute all consistent fusion rules for objects of those weights. There are several solutions, but every solution has at least one object satisfying the conditions of Lemma 4.6. Therefore $\mathcal{C} \cong \mathcal{AH}_2$, and the conclusion follows. \square

Theorem 4.9 *The fusion categories \mathcal{AH}_1 and \mathcal{AH}_2 do not admit any outer automorphisms.*

Proof Consider the algebra objects corresponding to the Asaeda-Haagerup subfactor in \mathcal{AH}_1 and \mathcal{AH}_2 . These algebra objects are all unique up to inner automorphism, so if any of these categories admit a non-trivial outer automorphism, we can find an outer automorphism which leaves one of these algebras invariant. Since each such algebra object admits a unique algebra structure by 3-supertransitivity, the outer automorphism must act trivially on the algebra. Hence it is enough to show that the Asaeda-Haagerup subfactor planar algebra does not admit any outer automorphism. Since there are no other subfactors of index $\frac{5+\sqrt{17}}{2}$, this planar algebra is generated by a single 6-box which is a rotational eigenvector and satisfies a certain quadratic equation (see [Jon03]). Any outer automorphism would send this rotational eigenvector to a multiple of itself, and the only multiple which satisfies the quadratic equation is itself. Hence there are no outer automorphisms. \square

As we will see later \mathcal{AH}_3 also has no outer automorphisms. This is not terribly difficult to show directly, but such a direct argument is more tedious since we can't use uniqueness of a subfactor of that specific index.

The three categories $\mathcal{AH}_1, \mathcal{AH}_2, \mathcal{AH}_3$ will be called the Asaeda-Haagerup fusion categories.

5 The combinatorics of Brauer-Picard groupoids

We first summarize the key ideas of the next two sections, and then give rigorous definitions and statements.

Any time you have a category, you can *decategorify* by turning isomorphisms into equalities (i.e. taking the Grothendieck group). For example, fusion categories decategorify to give fusion rings. In general you lose a lot of information passing from a category to its decategorification. For example, different fusion

categories can give the same fusion ring, and some fusion rings come from no fusion categories. However, in many cases you can prove results only by looking at the combinatorics of fusion rings (see [Izu91, Bis97] for some examples). Decategorifying the whole Brauer-Picard groupoid yields a bunch of structures. Each fusion category gives a fusion ring, each bimodule category yields a fusion bimodule, and the composition gives a composition rule for fusion bimodules. These structures satisfy a bunch of compatibility conditions.

Perhaps the most interesting part of this structure is the composition rules. Given two bimodule categories ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ and ${}_{\mathcal{D}}\mathcal{N}_{\mathcal{E}}$ there's a well-defined composition, the Deligne tensor product ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}} \boxtimes_{\mathcal{D}} {}_{\mathcal{D}}\mathcal{N}_{\mathcal{E}}$. In particular, given objects $m \in M$ and $n \in N$, we get an object mn in the tensor product. Taking Grothendieck groups, we have two bimodules ${}_{K(\mathcal{C})}K(\mathcal{M})_{K(\mathcal{D})}$ and ${}_{K(\mathcal{D})}K(\mathcal{N})_{K(\mathcal{E})}$, but their tensor product is typically not their tensor product as bimodules. In fact, knowing the bimodule structures on $K(\mathcal{M})$ and $K(\mathcal{N})$ is not enough to determine $K(\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N})$ even as an abelian group! Instead the categorical structure yields extra information: a composition map $K(\mathcal{M}) \otimes_{K(\mathcal{D})} K(\mathcal{N}) \rightarrow K(\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N})$. Perhaps surprisingly, in our case there are usually not very many possible compositions.

By considering the whole decategorified Brauer-Picard groupoid at once, we can often rule out possible fusion rings or fusion bimodules which look fine locally. For example, suppose we have a candidate fusion bimodule M over two fusion rings A and B , and further suppose we have a known bimodule category which decategorifies to give a fusion bimodule ${}_B N_C$. If M can be realized by an actual bimodule category, then it follows that there must be some composition rule sending $M \otimes_B N$ to a valid bimodule between M and N . If there are no such composition rules then we can rule out M as coming from an actual bimodule category.

5.1 Fusion modules, fusion bimodules, and multiplication maps

Any semisimple module category over a fusion category induces a representation of the Grothendieck ring of the fusion category on the Grothendieck group of the module category. We first recall the Frobenius-Perron theory of fusion rings and modules. For more details, see [ENO05].

Definition 5.1 A *fusion ring* (F, S) is a ring F whose additive group is the free Abelian group on a finite set $1 \in S$ which is equipped with an involution denoted by $\xi \mapsto \bar{\xi}$, $\xi \in S$ and whose multiplication is transitive with respect to

the basis S and satisfies $\xi\eta = \sum_{\mu \in S} N_{\eta\mu}^{\xi}$ for all $\xi, \eta \in S$, where the $N_{\eta\mu}^{\xi}$ are non-negative integers such that $N_{\eta\mu}^{\xi} = N_{\bar{\eta}\xi}^{\mu} = N_{\mu\eta}^{\bar{\xi}}$. The Frobenius-Perron dimension of an element of F is its image under the unique nonzero homomorphism $F \mapsto \mathbb{R}$ which maps S into the set of positive numbers.

Here the basis S and the involution are considered part of the data of the fusion ring. We will call the elements of the basis S the “simple vectors”. The Grothendieck ring of a fusion category is always a fusion ring, but there are many fusion rings which do not arise as the Grothendieck ring of any fusion category. We are interested in representations of fusion rings which respect the fusion structure.

Definition 5.2 A (left) fusion module of a fusion ring (F, S) on a finite set T is an indecomposable (left) representation of the fusion ring as endomorphisms of the free Abelian group on T such that the action satisfies $\xi\eta = \sum_{\mu \in T} N_{\eta\mu}^{\xi}$ for all

$\xi \in S, \eta, \mu \in T$, where the $N_{\eta\mu}^{\xi}$ are non-negative integers such that $N_{\eta\mu}^{\xi} = N_{\mu\eta}^{\bar{\xi}}$ for all ξ, μ, η . A Frobenius-Perron dimension vector is a positive real vector indexed by T which is an eigenvector for all the matrices $N^{\xi}, \xi \in S$. Such a dimension vector exists and is unique up to scalar multiples.

Remark 5.3 Fusion modules are sometimes called *nimreps*, which stands for non-negative integer matrix representations.

Again the basis of the free Abelian group T is considered part of the data of the fusion module, and again we will call those basis elements simple vectors. In a similar way, we can define right fusion modules and fusion bimodules. There are obvious notions of isomorphisms of fusion rings, fusion modules, and fusion bimodules (namely there should be bijections on the basis sets which preserve the algebraic structure).

Given a fusion module of (F, S) on T , we denote by (μ, η) the dot product of two elements of $\mathbb{Z}[T]$ with respect to the basis T , and similarly for two elements of F (with respect to the basis S .) We also define *right multiplication by duals* for a left fusion module as follows: $\mu\bar{\eta} := \sum_{\xi \in S} (\mu, \xi\eta)\xi$. Note that $\bar{\eta}$ is not actually an element of the fusion module; the expression is just a formal argument for multiplication. Similarly for a right fusion module we may define left multiplication by duals.

Definition 5.4 A left fusion module (M, T) of a fusion ring (F, S) is called *admissible* if there is a normalization of the Frobenius-Perron dimension vector d_M such that for every $\mu \in T$, we have:

- (a) $d_F(\mu\bar{\mu}) = d_M(\mu)^2$, where d_F is the Frobenius-Perron dimension vector for M .
- (b) $(\xi\mu, \eta\mu) = (\xi(\mu\bar{\mu}), \eta)$ for all $\xi, \eta \in S$.

Similarly, we can define admissible right fusion modules. A fusion bimodule (M, T) is admissible if its component left and right modules are admissible with the same normalized Frobenius-Perron dimension vectors and for any $\mu, \nu \in T$, we have:

$$(\mu\bar{\nu}, \mu\bar{\nu}) = (\bar{\mu}\mu, \bar{\nu}\nu) \text{ and } (\bar{\mu}\nu, \bar{\mu}\nu) = (\mu\bar{\mu}, \nu\bar{\nu}).$$

Lemma 5.5 Any fusion module arising from a module category over a fusion category is admissible. Any fusion bimodule arising from a bimodule category over a pair of fusion categories is admissible.

Proof For the first part note that $\mu\bar{\mu}$ is the element of the fusion ring corresponding to the internal endomorphisms of the object in the module category to which μ corresponds. The second part follows from Frobenius reciprocity for bimodule categories. \square

Suppose $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are fusion categories with fusion rings A, B, C and

${}_{\mathcal{A}}\mathcal{K}_{\mathcal{B}}, {}_{\mathcal{B}}\mathcal{L}_{\mathcal{C}}, {}_{\mathcal{A}}\mathcal{M}_{\mathcal{C}}$ are bimodule categories with fusion bimodules

${}_AK_B, {}_BL_C, {}_AM_C$ such that $\mathcal{K} \otimes_{\mathcal{B}} \mathcal{L} \cong \mathcal{M}$. The Frobenius-Perron dimensions of the fusion bimodules determine a dimension function on the relative tensor product of K and L which is multiplicative in the tensor factors, and the equivalence of categories induces a map from $K \otimes_B L$ to M which preserves the Frobenius-Perron dimension. Moreover, this map preserves multiplication by duals in the following sense: if $\xi_1 \otimes \eta_1 \mapsto \sum a_i \mu_i$ and $\xi_2 \otimes \eta_2 \mapsto \sum b_i \mu_i$ where $\xi_1, \xi_2 \in K$, $\eta_1, \eta_2 \in L$, are basis elements and the sums are over the basis $\mu_i \in M$, then for any $\lambda \in A$ we have $(\xi_1(\eta_1\bar{\eta}_2)\bar{\xi}_2, \lambda) = \sum a_i b_j (\mu_i \bar{\mu}_j, \lambda)$ and for any $\sigma \in C$ we have $(\bar{\xi}_1(\bar{\eta}_1\eta_2)\xi_2, \sigma) = \sum a_i b_j (\bar{\mu}_i \mu_j, \sigma)$.

Definition 5.6 (a) A *multiplication map* on a triple of admissible fusion bimodules $({}_AK_B, {}_BL_C, {}_AM_C)$ is a homomorphism from ${}_AK \otimes_B {}_LC$ to ${}_AM_C$ which preserves the Frobenius-Perron dimension functions and preserves multiplication by duals. The triple is (K, L, M) said to be *multiplicatively compatible* if there exists such a multiplication map.

(b) Similarly, a multiplication map on a triple of admissible fusion modules/bimodules $(K_A, {}_A L_B, M_B)$ is a homomorphism from $K \otimes_A L_B$ to M_B which preserves the Frobenius-Perron dimension functions and preserves right multiplication by duals. The triple (K, L, M) is said to be *multiplicatively compatible* if there exists such a multiplication map.

Note that in general such a multiplication map, if it exists, is not uniquely determined by the fusion structures.

5.2 Algorithms for computation

In this section we give some algorithms for finding the admissible fusion modules for a fusion ring, the admissible bimodules for two fusion rings, and the multiplication maps for triples of fusion modules or bimodules. It is easy to see that these are all finite problems. Indeed we are always looking for a finite list of natural numbers, and it is not difficult to see via dimension considerations that all of these natural numbers are bounded by the global dimension of the fusion category. Thus in principle one could simply enumerate all possibilities and check which work. In practice this is a hopelessly long computation, even for very simple fusion rings, so we give some more efficient algorithms. These algorithms are still only practical for small fusion rings but are sufficient for the Asaeda-Haagerup rings.

First, we describe a simple algorithm for finding all the decompositions of a positive semi-definite square non-negative integer matrix M into AA^T for non-negative integer matrix A with no zero columns (up to permutations of columns of A).

Definition 5.7 The set of sum of squares decompositions of a positive integer N is the set of all vectors of pairs of positive integers $(a_i, b_i), 1 \leq i \leq r$ such that $a_i < a_j$ whenever $i < j$ and $\sum_{i=1}^r b_i a_i^2 = N$.

Definition 5.8 An m -partial decomposition of a positive semi-definite $n \times n$ non-negative integer matrix M is an $m \times k$ non-negative integer matrix P , $0 \leq m \leq n$, $0 \leq k$, such that P has no zero columns and $P_i \cdot P_j = M_{ij}$ for all $1 \leq i, j \leq m$. We define $P_i \cdot P_j = 0$ if $k = 0$.

Thus the problem is to find all n -partial decompositions of M . The algorithm proceeds by describing all $(m+1)$ -partial decompositions whose first m rows form a given m -partial decomposition.

Algorithm 5.9 To find all decompositions of a positive semi-definite $n \times n$ non-negative integer matrix M into AA^T for a non-negative integer matrix A with no zero columns (up to permutations of columns of A):

- 1) Find the sets of sum of squares decompositions of all diagonal entries of M .
- 2) Start with the unique 0-partial decomposition of M (the empty matrix). Then for every m -partial decomposition P of M , $m < n$, consider all $(m+1)$ -partial decompositions as follows:

Case 1: P is empty, $M_{(m+1)(m+1)} = 0$. By positive semi-definiteness of M , the unique $(m+1)$ -partial decomposition is empty as well.

Case 2: P is empty, $M_{(m+1)(m+1)} = N > 0$. Then for each sum of squares decomposition of N , (a_i, b_i) , $1 \leq i \leq r$ we get an $(m+1)$ -partial decomposition by taking a vector with b_i copies of a_i for each i , and then adding m rows of zeros above it to complete the matrix with $m+1$ rows.

Case 3: P has k columns, $k > 0$, $M_{(m+1)(m+1)} = 0$. Then adding a row of zeros to P gives the unique $m+1$ -partial decomposition which extends P .

Case 4: P has k columns, $k > 0$, $M_{(m+1)(m+1)} = N > 0$. Then $(m+1)$ -partial decompositions which extend P correspond to the following data: a sum of squares decomposition of N , (a_i, b_i) , $1 \leq i \leq r$, and a vector $v = (v_1, \dots, v_k)$ such that:

- (a) $v_j \in \{a_i | 1 \leq i \leq r\} \cup \{0\}$ for all $1 \leq j \leq k$
- (b) $|\{j | v_j = a_i\}| \leq b_i$ for all $1 \leq i \leq r$
- (c) $v \cdot P_l = M_{l(m+1)}$ for all $1 \leq l \leq m$.

Given such data, we can form an $(m+1)$ -partial decomposition by adding a row to P with values equal to v , and then adding a column for each “leftover” member of the sum of squares decomposition with zeros above.

We can use the above decomposition algorithm to find all possible principal graphs of simple objects in simple module categories over a given fusion category following the technique from [GS11, §3].

Algorithm 5.10 To find all possible fusion matrices of simple vectors in admissible fusion modules over a given fusion ring (F, S) :

- 1) Compute the fusion matrix for each $\xi \in F$ such that $(\xi, 1) = 1$ and $0 \leq (\xi, \eta) \leq d(\eta)$ for all $\eta \in S$, and check whether it is positive semi-definite.
- 2) For each such positive semi-definite fusion matrix M^ξ , compute the set of decompositions $M^\xi = AA^T$ for A a non-negative integer matrix.

Given a decomposition of a fusion matrix $M^\xi = AA^T$ we assign dimensions to the columns of A through the vector $d_A = \frac{1}{\sqrt{d(\xi)}} A^T d$, where d is the Frobenius-Perron dimension vector of F . We define the dimension of A to be $d(A) := \sqrt{d(\xi)}$.

We can now use this list of possible principal graphs to find possible module categories.

Algorithm 5.11 *To find all admissible fusion modules over a fusion ring (F, S) :*

- 1) Find all possible fusion matrices for the simple vectors, i.e. non-negative integer matrices A such that $AA^T = M^\xi$ as in the previous algorithm. Sort the columns of each fusion matrix A by dimension.
- 2) For each n , find all n -tuples of fusion matrices (A_1, \dots, A_n) such that all the A_i have n columns and share a common column dimension vector d_A with $d(A_i) = d_A(i)$.
- 3) For each n -tuple (s_1, \dots, s_n) of elements of the symmetric group S_n such that $s_i(d_A) = d_A$ under the action of permuting the entries for all i , check whether the n -tuple $(s_1(A_1), \dots, s_n(A_n))$, where the s_i act on the A_i by permuting columns, gives consistent structure constants for a fusion module over F .

In Step 3 it is impractical to actually consider all n -tuples of permutations at once; rather we inductively find the consistent n -tuples of permutations by starting with s_1 and adding one permutation at a time, checking consistency at each step as in the previous algorithms.

Once we have the modules finding the bimodules is easy:

Algorithm 5.12 *To find the admissible bimodules over a pair of fusion rings F, G :*

- 1) Find all pairs (M, N) such that M is an admissible left fusion module over F and N is an admissible right fusion module over G and such that M and N have the same dimension vector. Fix an ordering of the bases of M and N .
- 2) For each such pair and each dimension preserving permutation s of the basis of M , check whether the left action of F commutes with the right action of G when the bases of M and N are identified by the permutation s , and if so, check whether the resulting bimodule satisfies the admissibility condition for fusion bimodules.

Let $({}_AK_B, {}_BL_C, {}_AM_C)$ be a triple of fusion bimodules, with bases ξ_i for $1 \leq i \leq l$, η_j for $1 \leq j \leq m$, and μ_k for $1 \leq k \leq n$, respectively.

Definition 5.13 A (p, q) -partial multiplication map is an assignment of a vector of length n , v_k^{ij} , for each pair (i, j) , $1 \leq i \leq p$, $1 \leq j \leq q$ such that:

- (a) $d(\xi_i)d(\eta_j) = \sum_{k=1}^n v_k^{ij} d(\mu_k)$
- (b) $(\bar{x}_i x_i, y_j \bar{y}_j) = \sum_{k=1}^n (v_k^{ij})^2$ for all $1 \leq i_1, i_2 \leq p, 1 \leq j_1, j_2 \leq q$
- (c) for all $\lambda \in A, 1 \leq i_1, i_2 \leq p, 1 \leq j_1, j_2 \leq q$ we have $(\xi_{i_1}(\eta_{j_1} \bar{\eta}_{j_2}) \bar{\xi}_{i_2}, \lambda) = \sum v_{k_1}^{i_1 j_1} v_{k_2}^{i_2 j_2} (\mu_{k_1} \bar{\mu}_{k_2}, \lambda)$.
- (c') for all $\lambda \in C, 1 \leq i_1, i_2 \leq p, 1 \leq j_1, j_2 \leq q$ we have $(\bar{\xi}_{i_1}(\eta_{j_1} \eta_{j_2}) \xi_{i_2}, \lambda) = \sum v_{k_1}^{i_1 j_1} v_{k_2}^{i_2 j_2} (\bar{\mu}_{k_1} \mu_{k_2}, \lambda)$.

Algorithm 5.14 To check whether a given triple of admissible fusion bimodules is multiplicatively compatible:

Start with a $(0, 0)$ -partial multiplication. Then find all extensions of a given (p, q) -partial multiplication map as follows:

Case 1: $p < l$. Find candidates for $v^{(p+1), q}$ by checking conditions (a) and (b) above as follows: for each sum of squares decomposition (a_i, b_i) of $(\bar{\xi}_{p+1} \xi_{p+1}, \eta_q \bar{\eta}_q)$ such that $\sum b_i \leq n$, form the vector v of size n given by b_i copies of each a_i with the rest of the entries equal to 0. Then for each permutation $s \in S_n$, check whether $s(v).d = d(\xi_i)d(\eta_j)$, where d is the dimension vector of the bimodule C . Finally, if conditions (a) and (b) are satisfied for a candidate vector $v' = \sigma(v)$, check conditions (c), (c') for $v^{(p+1), q} = v'$.

Case 2: $p = l, q < m$. Same as previous case but with the bi-index $(0, q+1)$ instead of $(p+1, q)$.

Case 3: $p = l, q = m$. Check whether the given map commutes appropriately with the actions of A, B, C .

In a similar way, we can check whether a given right module/bimodule/right module triple is multiplicatively compatible, using only condition (c') and not (c).

6 The Brauer-Picard groupoid of the Asaeda-Haagerup categories

In this section we compute all Morita equivalences between the three Asaeda-Haagerup categories and prove several results concerning the full Brauer-Picard groupoid.

6.1 Fusion modules and bimodules of the Asaeda-Haagerup fusion rings

The Grothendieck rings of the Asaeda-Haagerup fusion categories will be called the Asaeda-Haagerup fusion rings.

Theorem 6.1 (a) *Up to isomorphism, the admissible fusion modules over the Asaeda-Haagerup fusion rings are classified as follows: there are 24 AH1-modules, 21 AH2-modules, and 20 AH3-modules. Full multiplication rules for all these modules are included in the supplementary files AH1Modules, AH2Modules, and AH3Modules.*

(b) *Up to isomorphism, the admissible fusion bimodules over the Asaeda-Haagerup fusion rings are classified as follows: there are 14 AH1 – AH1-bimodules, 13 AH2 – AH2-bimodules, 13 AH3 – AH3 bimodules, 9 AH1 – AH2 bimodules, 7 AH1 – AH3 bimodules, and 6 AH2 – AH3 bimodules. Full multiplication rules for all these bimodules are included in the supplementary file Bimodules.*

Proof Apply the algorithms from the previous section. □

For all pairs (x, y) where x is an admissible fusion $AHi - AHj$ -bimodule and y is an admissible fusion $AHj - AHk$ -bimodule for some $1 \leq i, j, k \leq 3$ we have computed the list of multiplicatively compatible triples (x, y, z) . These lists are included in the supplementary file *BimoduleCompatibility*.

Similarly, for all pairs (x, y) where x is an admissible fusion right AHi -module and y is an admissible fusion $AHi - AHj$ -bimodule for some $1 \leq i, j \leq 3$ we have computed the list of multiplicatively compatible triples (x, y, z) . These lists are included in supplementary file *ModuleCompatibility*.

6.2 Reductions using compatibility

We now analyze the Brauer-Picard groupoid using a sequence of deductions from known information and multiplicative compatibility.

We identify the fusion rings $AH1, AH2, AH3$ with the Grothendieck rings of the fusion categories $\mathcal{AH}_1, \mathcal{AH}_2, \mathcal{AH}_3$. We will say that an AHi fusion module (resp. $AHi - AHj$ fusion bimodule) is realized if it is induced by an \mathcal{AH}_i -module category (resp. $\mathcal{AH}_i - \mathcal{AH}_j$ -bimodule category.) We will say such a fusion module (resp. fusion bimodule) is realized uniquely if any two categories which realize it are equivalent as module categories (resp. bimodule categories). Any fusion bimodule has a dual bimodule; if the original fusion bimodule is realized by a bimodule category then the dual bimodule is realized by the opposite category.

We will be referring extensively to the lists of fusion modules, fusion bimodules, and multiplicatively compatible triples over the Asaeda-Haagerup fusion rings. These lists are given in the supplementary data files *AH1Modules*, *AH2Modules*, *AH3Modules*, *Bimodules*, *BimoduleCompatibility*, and *Module Compatibility*.

We introduce the following notation. The symbol a_i will denote the a^{th} fusion module on the list of AHi fusion modules in the supplementary file *AH1Modules*, *AH2Modules*, or *AH3Modules* (depending on whether $i = 1, 2$, or 3). Similarly, a_{ij} will denote the a^{th} fusion bimodule on the list of $AHi - AHj$ fusion bimodules. For two fusion bimodules a_{ij} and b_{jk} , $a_{ij} \cdot b_{jk}$ will denote the set of bimodules z_{ik} such that the triple (a_{ij}, b_{jk}, z_{ik}) appears on the list of triples which are multiplicatively compatible. If there is a unique such c_{ik} , we say that a_{ij} and b_{jk} have a unique multiplication and write $a_{ij}b_{jk} = c_{ik}$. The same notation is also used when a_i is a fusion module and b_{ij} is a fusion bimodule.

Lemma 6.2 *The following fusion bimodules are realized:*

$12_{11}, 14_{11}, 9_{12}, 6_{13}, 9_{21}, 13_{22}, 6_{23}, 6_{31}, 6_{32}, 13_{33}$.

Proof The bimodules $14_{11}, 13_{22}, 13_{33}$ are realized by the trivial autoequivalences of $\mathcal{AH}_1, \mathcal{AH}_2, \mathcal{AH}_3$, respectively. The bimodules $9_{12}, 6_{13}, 12_{11}$ are realized by the AH , $AH + 1$, and $AH + 2$ subfactors, respectively. The bimodule 6_{23} is realized by the two-dimensional algebra objects in \mathcal{AH}_2 and \mathcal{AH}_3 by Theorem 4.8. The bimodules $9_{21}, 6_{31}, 6_{32}$ are the dual bimodules to $9_{12}, 6_{13}, 6_{23}$. \square

Lemma 6.3 *The following fusion bimodules are realized:*

$8_{11}, 13_{11}, 2_{12}, 5_{12}, 8_{12}, 2_{13}, 3_{13}, 7_{13}, 2_{21}, 5_{21}, 8_{21}, 8_{22}, 12_{22}, 1_{23}, 2_{31}, 3_{31}, 7_{31}, 1_{32}, 11_{33}, 12_{33}$.

Proof If two fusion bimodules a_{ij} and b_{jk} are realized and we have a unique multiplication $a_{ij}b_{jk} = c_{ik}$ then c_{ik} is realized as well. We have the following unique multiplications (at each step we use the results of the previous step as inputs):

$$1) 12_{11}9_{12} = 8_{12}, 12_{11}6_{13} = 2_{13}, 9_{12}6_{23} = 7_{13}, 6_{13}6_{32} = 5_{12}.$$

$$2) 12_{11}5_{12} = 2_{12}, 12_{11}7_{13} = 3_{13}.$$

Note that the dual bimodules $2_{21}, 5_{21}, 8_{21}, 2_{31}, 3_{31}, 7_{31}$ are realized as well.

$$3) 2_{13}3_{31} = 13_{11}, 2_{21}5_{12} = 12_{22}, 5_{21}9_{12} = 8_{22}, 2_{21}7_{13} = 1_{23}, 2_{31}3_{13} = 12_{33}, 3_{31}7_{13} = 11_{33}$$

Again the dual bimodule 1_{32} is also realized.

$$4) 12_{11}13_{11} = 8_{11}.$$

□

Lemma 6.4 (a) Let $a_{ij}b_{jk} = c_{jk}$ be a unique multiplication. If a and c are realized uniquely and b is realized, then b is realized uniquely. The same result holds interchanging the roles of a and b or for a unique multiplication $a_ib_{ij} = c_j$.

(b) Let $a_{ij}\bar{a}_{ij} = Id_{ii}$ be a unique multiplication, where \bar{a}_{ij} is the dual fusion bimodule to a_{ij} , and Id_{ii} is the trivial bimodule for $AH_i, i = 1, 2$. Then if a_{ij} is realized, it is realized uniquely.

Proof (a) Suppose $a_{ij}b_{jk} = c_{jk}$ is a unique multiplication with a and c realized uniquely, and suppose that b is also realized. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be bimodule categories realizing a, b, c respectively. Then we have $\mathcal{A} \otimes \mathcal{B} \cong \mathcal{C}$, so $\mathcal{B} \cong \mathcal{A}^{-1} \otimes \mathcal{C}$; since a, c are uniquely realized, \mathcal{A}, \mathcal{C} are uniquely determined, and therefore so is \mathcal{B} . The proofs of the other statements are similar.

(b) Suppose $a_{ij}\bar{a}_{ij} = Id_{ii}$ is a unique multiplication, where $i = 1, 2$. Since $\mathcal{AH}_1, \mathcal{AH}_2$ have no outer automorphisms (Theorem 4.9), Id_{ii} is realized uniquely. Suppose $\mathcal{A}_1, \mathcal{A}_2$ are two realizations of a . Then by the unique multiplication of $a\bar{a}$ and the unique realization of Id_{ii} we have $\mathcal{A}_1 \otimes (\mathcal{A}_2)^{-1} \cong \mathcal{I}_i$, where \mathcal{I}_i is the trivial auto-equivalence of \mathcal{AH}_i ; therefore we have $\mathcal{A}_1 \cong \mathcal{A}_2$. So if a is realized it must be realized uniquely. □

Lemma 6.5 The following fusion bimodules are realized: $11_{22}, 8_{33}$.

Proof We know that 8_{22} and 12_{22} are realized. Moreover, since $8_{22}8_{22} = 13_{22}$ is a unique multiplication, 8_{22} is realized uniquely. Therefore at least one member of $8_{22} \cdot 12_{22} = \{8_{22}, 9_{22}, 10_{22}, 11_{22}\}$ distinct from 8_{22} must be realized. However, 9_{22} cannot be realized since $9_{22} \cdot 8_{22}$ is empty, and 10_{22} cannot be realized since $8_{22} \cdot 10_{22}$ is empty. Therefore 11_{22} is realized.

Similarly, we know that 11_{33} and 12_{33} are realized, and since $11_{33}11_{33} = 13_{33}$ is a unique multiplication, 11_{33} is realized uniquely. Therefore at least one member of $11_{33} \cdot 12_{33} = \{8_{33}, 9_{33}, 10_{33}, 11_{33}\}$ distinct from 11_{33} must be realized. Since $10_{33} \cdot 11_{33}$ and $11_{33} \cdot 9_{33}$ are empty, 8_{33} must be realized. \square

Lemma 6.6 *The fusion bimodules $1_{23}, 3_{23}, 4_{23}, 6_{23}$, and their duals*

$1_{32}, 3_{32}, 4_{32}, 6_{32}$ are realized uniquely. The bimodules 2_{23} and 5_{23} are not realized.

Proof We already know that 1_{23} and 6_{23} are realized. Since 5_{12} is realized and $5_{12}5_{23}$ is empty, 5_{23} cannot be realized. Since 3_{13} is realized $3_{13}2_{32}$ is empty, 2_{32} and its dual 2_{23} cannot be realized. Since 5_{21} and 7_{13} are realized and $5_{21} \cdot 7_{13} = \{2_{23}, 3_{23}\}$, and 2_{23} is not realized, 3_{23} must be realized. Since 8_{21} and 7_{13} are realized and $8_{21} \cdot 7_{13} = \{4_{23}, 5_{23}\}$, and 5_{23} is not realized, 4_{23} must be realized.

Uniqueness of the realizations of $3_{23}, 4_{23}, 6_{23}$ follows from Lemma 6.4 (b). For uniqueness of the realization of 1_{23} we must use Lemma 6.4 several times. First, 7_{13} and its dual 7_{31} are uniquely realized by Lemma 6.4 (b). Second, we have a unique multiplication $12_{11}2_{12} = 5_{12}$. Third, we have $2_{12} \cdot 2_{21} = \{12_{11}, 14_{11}\}$. However, given a pair of bimodule categories realizing 2_{12} and 2_{21} , their tensor product must realize 14_{11} and not 12_{11} , since $2_{12} \notin 12_{11} \cdot 2_{12}$. Therefore the same argument as in Lemma 6.4 (b) applies, and 2_{12} and 2_{21} are uniquely realized. Finally, we have the unique multiplication $1_{23}7_{31} = 2_{21}$, so since 2_{21} and 7_{31} are uniquely realized, by Lemma 6.4 (a), 1_{23} is uniquely realized as well. \square

Theorem 6.7 (a) *There are exactly 4 bimodule categories over each not-necessarily-distinct pair $\mathcal{A}\mathcal{H}_i - \mathcal{A}\mathcal{H}_j$, up to equivalence. These realize the following fusion bimodules, which are each realized uniquely:*

$8_{11}, 12_{11}, 13_{11}, 14_{11}, 2_{12}, 5_{12}, 8_{12}, 9_{12}, 2_{13}, 3_{13}, 6_{13}, 7_{13}$

$2_{21}, 5_{21}, 8_{21}, 9_{21}, 8_{22}, 11_{22}, 12_{22}, 13_{22}, 1_{23}, 3_{23}, 4_{23}, 6_{23}$

$2_{31}, 3_{31}, 6_{31}, 7_{31}, 1_{32}, 3_{32}, 4_{32}, 6_{32}, 8_{33}, 11_{33}, 12_{33}, 13_{33}.$

(b) *The Brauer-Picard group of each $\mathcal{A}\mathcal{H}_i$ is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.*

Proof (a) We have already seen that all of these bimodules are realized. Since by Lemma 6.6 there are exactly 4 bimodule categories between \mathcal{AH}_2 and \mathcal{AH}_3 , there must be exactly 4 bimodule categories between each pair $\mathcal{AH}_i - \mathcal{AH}_j$. Therefore each of the bimodule categories on the list must be realized uniquely and there can be no others.

(b) This is immediate from (a) and the unique multiplications $8_{11}8_{11} = 12_{11}12_{11} = 13_{11}13_{11} = 14_{11}$. \square

Corollary 6.8 \mathcal{AH}_3 does not admit any outer automorphisms.

Proof The Brauer-Picard group has order 4 and we know that 4 distinct fusion bimodules (which are also different as fusion modules) are realized by auto-equivalences; therefore there cannot be any outer automorphisms. \square

Lemma 6.9 (a) The fusion modules $2_1, 3_1, 5_1, 7_1, 8_1, 12_1, 13_1, 14_1, 15_1, 22_1, 23_1, 24_1$ are each realized uniquely. The fusion modules $4_1, 6_1, 9_1, 10_1, 17_1, 18_1, 20_1, 21_1$ are not realized.

(b) The fusion modules $1_2, 2_2, 5_2, 6_2, 7_2, 8_2, 10_2, 12_2, 14_2, 16_2, 20_2, 21_2$ are each realized uniquely. The fusion modules $9_2, 11_2, 13_2, 15_2, 18_2$ are not realized.

(c) The fusion modules $1_3, 4_3, 5_3, 6_3, 7_3, 8_3, 10_3, 12_3, 15_3, 17_3, 19_3, 20_3$ are each realized uniquely. The fusion modules $9_3, 11_3, 13_3, 14_3$ are not realized.

Proof We prove (c). The first list of 12 modules are realized by the 12 $\mathcal{AH}_i - \mathcal{AH}_3$ bimodule categories above. For uniqueness, we use Lemma 6.4. First note that the trivial modules $22_1, 21_2, 20_3$ are realized uniquely. Also, the modules $12_1, 13_1, 23_1, 14_2, 16_2, 8_3, 15_3$ are realized uniquely by the uniqueness of the $AH, AH + 1, AH + 2$ and dimension 2 algebra objects.

We have the following unique multiplications: $5_3 3_{31} = 22_1, 6_3 3_{31} = 12_1, 7_3 11_{33} = 20_3, 10_3 6_{31} = 23_1, 12_3 4_{32} = 21_2, 17_3 7_{31} = 22_1, 19_3 7_{31} = 12_1$, which proves the unique realization of $5_3, 6_3, 7_3, 10_3, 12_3, 17_3, 19_3$. That leaves 1_3 and 4_3 .

The modules 6_2 and 7_2 are realized with the bimodules 8_{22} and 11_{22} . We have unique multiplications $6_2 5_{21} = 23_1$ and $7_2 11_{22} = 21_2$, so the modules are realized uniquely. Since $1_3 \cdot 6_{32} = \{6_2, 7_2\}$, this implies that any realization of 1_3 must have dual category \mathcal{AH}_2 . But we already know that 1_{23} , which is the only $AH_2 - AH_3$ bimodule extension of 1_3 , is realized uniquely, so 1_3 is realized uniquely.

Similarly, the module 2_2 is realized by the bimodule 2_{12} , and by the unique multiplication $2_2 5_{21} = 13_1$ the realization is unique. Then from the unique multiplication $4_3 6_{32} = 2_2$, 4_3 is realized uniquely.

The proofs of (a) and (b) are similar, except easier since we can use the results of (c) when checking multiplicative compatibility.

□

We end with a classification of possible other objects in the groupoid.

Theorem 6.10 *Let \mathcal{C} be a fusion category which is Morita equivalent to the $\mathcal{AH}_i, i = 1, 2, 3$, but not equivalent to any of them. Then exactly one of the following for cases holds:*

- (a) Every $\mathcal{C} - \mathcal{AH}_1$ Morita equivalence realizes 11_1 , every $\mathcal{C} - \mathcal{AH}_2$ Morita equivalence realizes 19_2 , and every $\mathcal{C} - \mathcal{AH}_3$ Morita equivalence realizes 16_3 .
- (b) Every $\mathcal{C} - \mathcal{AH}_1$ Morita equivalence realizes 16_1 , every $\mathcal{C} - \mathcal{AH}_2$ Morita equivalence realizes 4_2 , and every $\mathcal{C} - \mathcal{AH}_3$ Morita equivalence realizes 18_3 .
- (c) Every $\mathcal{C} - \mathcal{AH}_1$ Morita equivalence realizes 19_1 , every $\mathcal{C} - \mathcal{AH}_2$ Morita equivalence realizes 17_2 , and every $\mathcal{C} - \mathcal{AH}_3$ Morita equivalence realizes 2_3 .
- (d) Every $\mathcal{C} - \mathcal{AH}_1$ Morita equivalence realizes 1_1 , every $\mathcal{C} - \mathcal{AH}_2$ Morita equivalence realizes 3_2 , and every $\mathcal{C} - \mathcal{AH}_3$ Morita equivalence realizes 3_3 .

Proof First note that by Lemma 6.9, the only modules whose realizations are not yet known are $1_1, 11_1, 16_1, 19_1, 3_2, 4_2, 17_2, 19_2, 2_3, 3_3, 16_3, 18_3$. Let \mathcal{C} be a fusion category which is Morita equivalent to the $\mathcal{AH}_i, i = 1, 2, 3$ but not equivalent to any of them. Then there is a $\mathcal{C} - \mathcal{AH}_3$ Morita equivalence which realizes one of the four modules $2_3, 3_3, 16_3, 18_3$. We will show that the four possibilities correspond to the four cases in the statement of the theorem.

First, suppose that a $\mathcal{C} - \mathcal{AH}_3$ Morita equivalence realizes 16_3 , and consider the action of the Brauer-Picard group of \mathcal{AH}_3 on the bimodule category. Since the Brauer-Picard group realizes the bimodules $8_{33}, 11_{33}, 12_{33}, 13_{33}$, the unique multiplications $16_3 8_{33} = 16_3, 16_3 12_{33} = 16_3, 16_3 13_{33} = 16_3$, together with the multiplication $16_3 \cdot 11_{33} = \{5_3, 16_3\}$, imply that every $\mathcal{C} - \mathcal{AH}_3$ Morita equivalence realizes 16_3 . Similarly, if a $\mathcal{C} - \mathcal{AH}_3$ Morita equivalence realizes 18_3 , every $\mathcal{C} - \mathcal{AH}_3$ Morita equivalence must realize 18_3 .

For 2_3 and 3_3 we have the following multiplications: $2_3 \cdot 8_{33} = \{2_3, 3_3, 18_3\}, 2_3 11_{33} = 2_3, 2_3 \cdot 12_{33} = \{2_3, 3_3\}, 2_3 13_{33} = 2_3, 3_3 \cdot 8_{33} = 3_3, 3_3 11_{33} = \{2_3, 3_3, 18_3\}, 3_3 \cdot 12_{33} =$

$\{2_3, 3_3\}, 3_3 13_{33} = 3_3, .$ From these possibilities, it is clear that if either 2_3 or 3_3 is realized by a $\mathcal{C} - \mathcal{AH}_3$ Morita equivalence then it is realized by multiple inequivalent $\mathcal{C} - \mathcal{AH}_3$ Morita equivalences. Therefore the only possibilities are either that 2_3 or 3_3 is realized by 4 different $\mathcal{C} - \mathcal{AH}_3$ Morita equivalences, or that 2 different $\mathcal{C} - \mathcal{AH}_3$ realize 2_3 and another 2 different $\mathcal{C} - \mathcal{AH}_3$ Morita equivalences realize 3_3 . Suppose the latter occurred. From the unique multiplication $2_3 11_{33} = 2_3$ we see that the nontrivial autoequivalence 11_{33} must permute the 2 realization of 2_3 . Similarly, from $3_3 \cdot 8_{33} = 3_3$ the nontrivial autoequivalence 8_{33} must permute the 2 realizations of 3_3 . But this means that 8_{33} and 11_{33} implement the same order 2 permutation on the set of $\mathcal{C} - \mathcal{AH}_3$ Morita equivalences, which implies that they are inverses of each other; since every autoequivalence in the Brauer group has order 2 this is impossible.

We have now seen that if any $\mathcal{C} - \mathcal{AH}_3$ Morita equivalence realizes one of $2_3, 3_3, 16_3, 18_3$, then every $\mathcal{C} - \mathcal{AH}_3$ realizes the same $AH3$ module. Similar arguments show the same thing for the 4 AH_1 modules and the 4 AH_2 modules in the statement of the theorem. It remains only to check which AH_i modules are compatible with which AH_j modules. This too can be sorted out from the multiplicative compatibility lists. If 16_3 is realized, then the unique multiplications $16_3 6_{32} = 19_2$ and $16_3 7_{31} = 11_1$ show that 19_2 and 11_1 must be realized as well.

The other three cases are handled similarly.

□

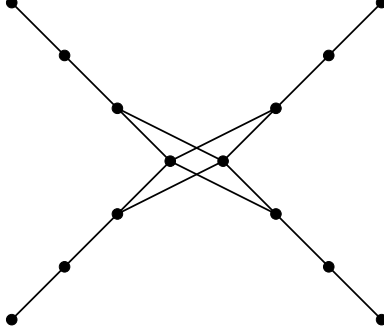
We can go quite a bit further in the classification of the four cases in this theorem. Specifically, we have the following conjecture.

Conjecture 6.11 Cases (a)-(c) above are each realized by a unique fusion category, each having the following properties:

- (a) There are 4 invertible objects and 4 objects of dimension $4 + \sqrt{17}$.
- (b) The Brauer-Picard group is implemented by outer automorphisms.

Note that property (b) is in sharp contrast to the situation with $\mathcal{AH}_1 - \mathcal{AH}_3$, which do not admit any outer automorphisms and whose Morita equivalences are also all distinct as module categories, and indeed, even as fusion modules.

We can reduce the proof of this conjecture to the construction of a single sub-factor with principal graph



We hope to address this in future work.

7 Subfactors in the Asaeda-Haagerup family

In this section we classify subfactors whose even parts are among the three Asaeda-Haagerup fusion categories, up to isomorphism of the planar algebra.

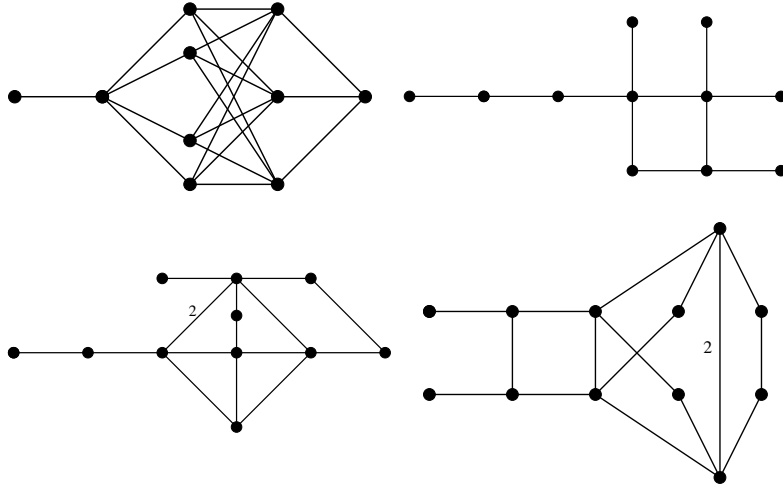
7.1 Principal graphs of small-index subfactors in the Asaeda-Haagerup categories

We now list the principal and dual graphs of a minimal dimension generating object in each of the 24 bimodule categories, which, up to duality and equivalence, exhaust the full subgroupoid of the Brauer-Picard groupoid generated by $\mathcal{AH}_1 - \mathcal{AH}_3$. By a generating object of a module category over a fusion category we mean an object whose internal endomorphisms tensor generates the fusion category. In this case of the Asaeda-Haagerup categories, this just excludes objects of dimensions 1 and $\sqrt{2}$, whose internal endomorphisms tensor generate trivial and $Vec_{\mathbb{Z}/2\mathbb{Z}}$ proper subcategories, respectively.

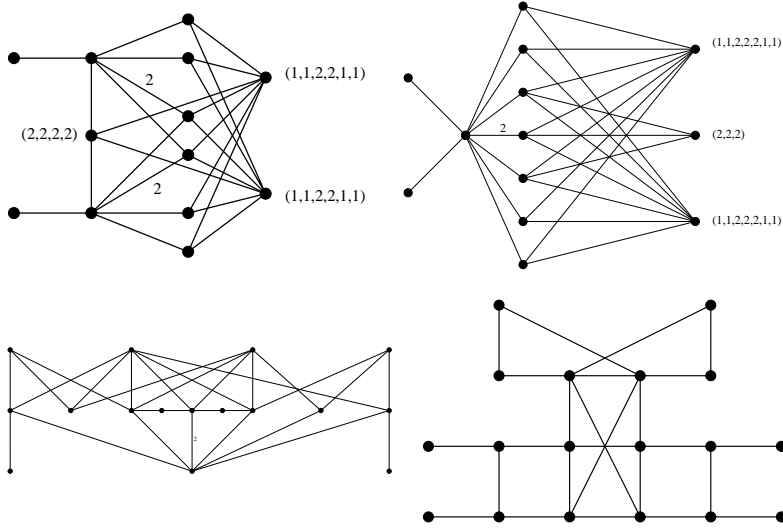
For the $\mathcal{AH}_i - \mathcal{AH}_i$ bimodule categories, the two graphs are always the same, so we only list the common graph once. For the $\mathcal{AH}_i - \mathcal{AH}_j, i \neq j$ bimodule categories, we give the ordered pair of principal graphs.

Multiplicities on edges are denoted by putting a number next to the edge, or, if the graph is too complex, putting a tuple of numbers next to a vertex. In the latter case you should read the numbers as labeling the edges from top to bottom.

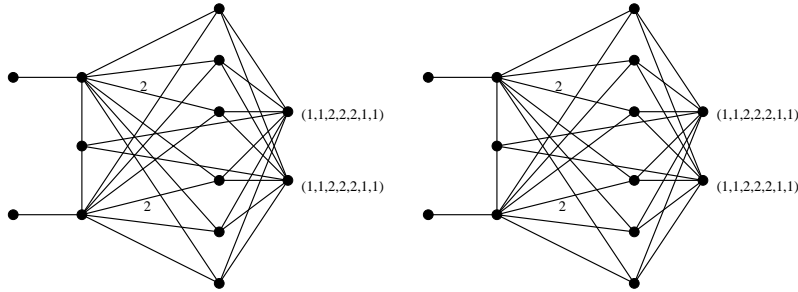
(a) $\mathcal{AH}_1 - \mathcal{AH}_1$ -categories:

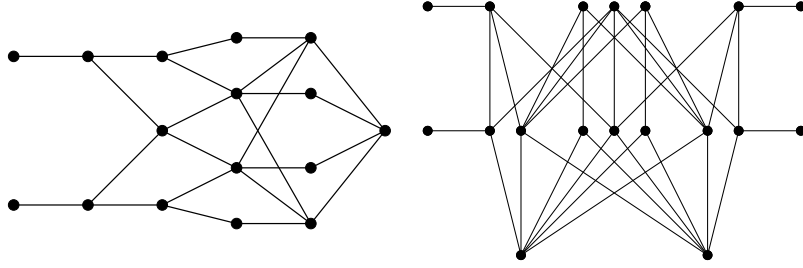


(b) $\mathcal{AH}_2 - \mathcal{AH}_2$ -categories:

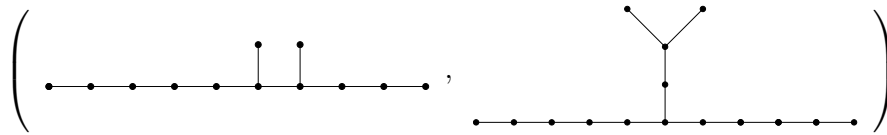
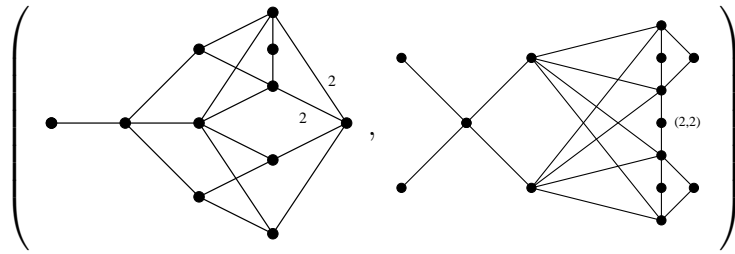
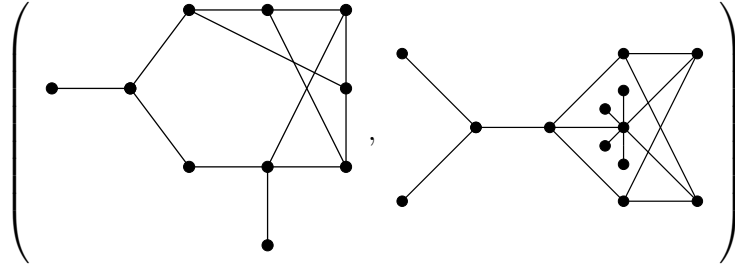
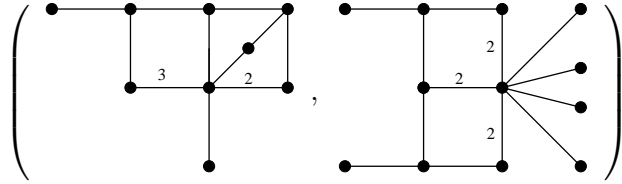


(c) $\mathcal{AH}_3 - \mathcal{AH}_3$ -categories:

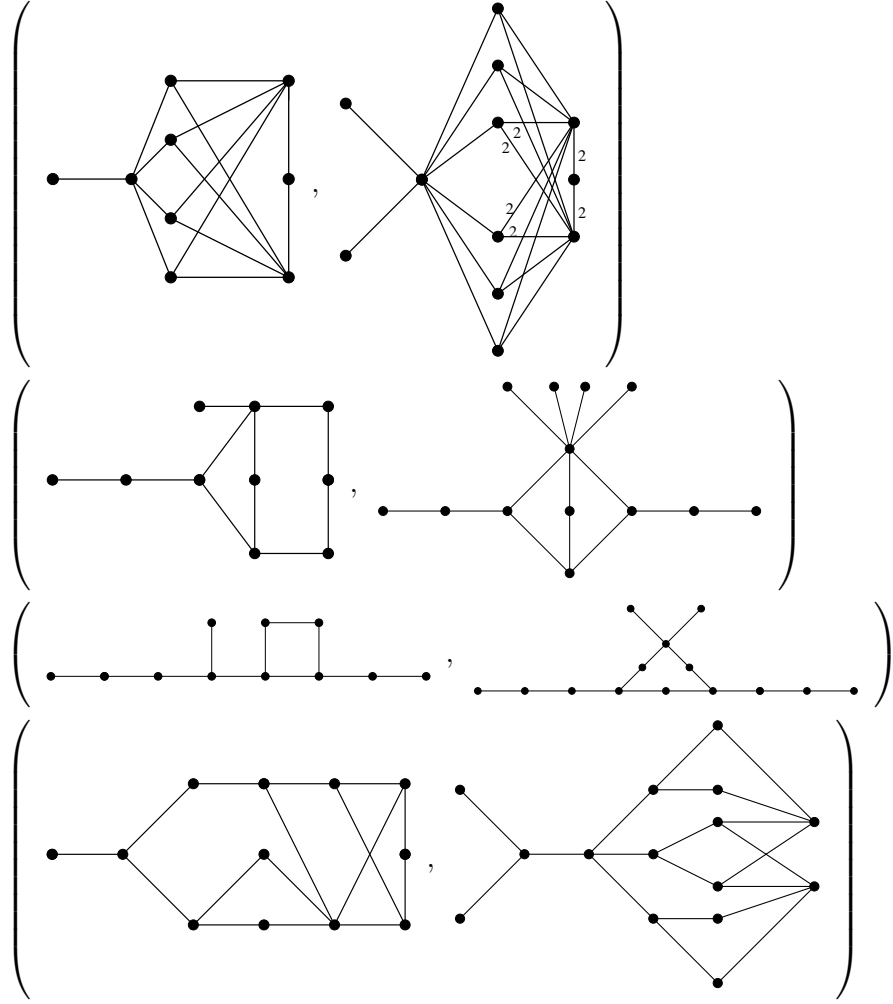




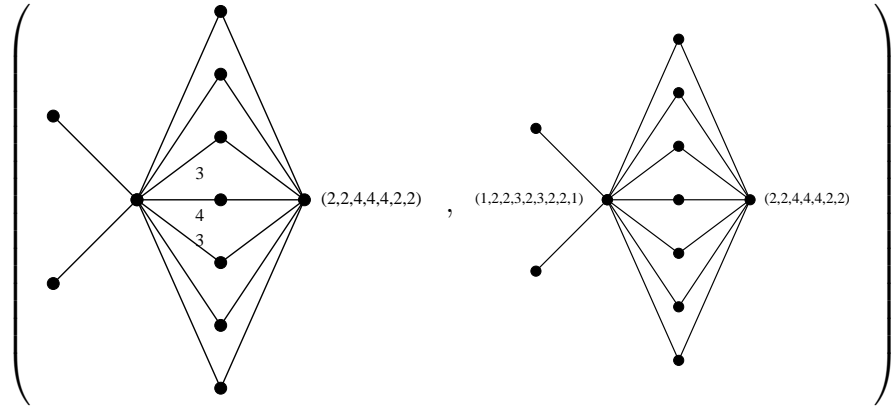
(d) $\mathcal{AH}_1 - \mathcal{AH}_2$ -categories:



(e) $\mathcal{AH}_1 - \mathcal{AH}_3$ -categories:



(f) $\mathcal{AH}_2 - \mathcal{AH}_3$ -categories:



even parts in $\{\mathcal{AH}_1, \mathcal{AH}_2, \mathcal{AH}_3\}$ arises this way. The question is when two different simple bimodule objects κ and λ give the same planar algebra, which happens iff there is an automorphism of the left fusion category which takes the algebra of internal endomorphisms of κ to that of λ . For $\{\mathcal{AH}_1, \mathcal{AH}_2, \mathcal{AH}_3\}$, the only nontrivial automorphisms are the unique inner automorphisms of \mathcal{AH}_2 and \mathcal{AH}_3 corresponding to the unique nontrivial invertible object in each case. So κ and λ give the same planar algebra iff there is an invertible object α in the left category such that $\alpha\kappa\bar{\alpha} \cong \lambda\bar{\lambda}$ (as algebra objects), which by Theorem 3.6 happens iff $\lambda \cong \alpha\kappa\beta$, where β is an object in the right category. But the action of invertible objects is known - it is contained in the fusion data. So the list of planar algebras can be read off the list of principal graphs above - there is a distinct planar algebra corresponding to each equivalence class of odd vertices, where equivalence is given by the combined left and right action of invertible objects.

□

References

- [AG11] Marta Asaeda and Pinhas Grossman. A quadrilateral in the Asaeda-Haagerup category. *Quantum Topol.*, 2(3):269–300, 2011.
- [AH99] Marta Asaeda and Uffe Haagerup. Exotic subfactors of finite depth with Jones indices $(5 + \sqrt{13})/2$ and $(5 + \sqrt{17})/2$. *Comm. Math. Phys.*, 202(1):1–63, 1999. MR1686551 DOI:10.1007/s002200050574 arXiv:math.OA/9803044.
- [BEK00] Jens Böckenhauer, David E. Evans, and Yasuyuki Kawahigashi. Chiral structure of modular invariants for subfactors. *Comm. Math. Phys.*, 210(3):733–784, 2000. MR1777347.
- [Bis97] Dietmar Bisch. Bimodules, higher relative commutants and the fusion algebra associated to a subfactor. In *Operator algebras and their applications (Waterloo, ON, 1994/1995)*, volume 13 of *Fields Inst. Commun.*, pages 13–63. Amer. Math. Soc., Providence, RI, 1997. MR1424954 (preview at google books).
- [DR89] Sergio Doplicher and John E. Roberts. A new duality theory for compact groups. *Invent. Math.*, 98(1):157–218, 1989.
- [EGNO] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. Tensor categories. Online.
- [EK98] David E. Evans and Yasuyuki Kawahigashi. *Quantum symmetries on operator algebras*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1998. MR1642584.
- [ENO05] Pavel Etingof, Dmitri Nikshych, and Viktor Ostrik. On fusion categories. *Ann. of Math. (2)*, 162(2):581–642, 2005. MR2183279 DOI:10.4007/annals.2005.162.581 arXiv:math.QA/0203060.

- [ENO10] Pavel Etingof, Dmitri Nikshych, and Victor Ostrik. Fusion categories and homotopy theory. *Quantum Topol.*, 1(3):209–273, 2010. With an appendix by Ehud Meir.
- [EO04] Pavel Etingof and Viktor Ostrik. Module categories over representations of $SL_q(2)$ and graphs. *Math. Res. Lett.*, 11(1):103–114, 2004. MR2046203 [arXiv:math.QA/0302130](#).
- [EP09a] David E. Evans and Mathew Pugh. Ocneanu cells and Boltzmann weights for the $SU(3)$ ADE graphs. *Münster J. Math.*, 2:95–142, 2009.
- [EP09b] David E. Evans and Mathew Pugh. $SU(3)$ -Goodman-de la Harpe-Jones subfactors and the realization of $su(3)$ modular invariants. *Rev. Math. Phys.*, 21(7):877–928, 2009.
- [GdlHJ89] Frederick M. Goodman, Pierre de la Harpe, and Vaughan F. R. Jones. *Coxeter graphs and towers of algebras*, volume 14 of *Mathematical Sciences Research Institute Publications*. Springer-Verlag, New York, 1989. MR999799.
- [GI08] Pinhas Grossman and Masaki Izumi. Classification of noncommuting quadrilaterals of factors. *Internat. J. Math.*, 19(5):557–643, 2008. [arXiv:0704.1121](#), MR2418197.
- [GS11] P. Grossman and N. Snyder. Quantum subgroups of the Haagerup fusion categories. *Comm. Math. Phys.*, 2011. [arXiv:1102.2631](#), accepted June 23 2011.
- [Izu91] Masaki Izumi. Application of fusion rules to classification of subfactors. *Publ. Res. Inst. Math. Sci.*, 27(6):953–994, 1991. MR1145672 DOI:10.2977/prims/1195169007.
- [Jon] Vaughan F. R. Jones. Planar algebras, I. [arXiv:math.QA/9909027](#).
- [Jon83] Vaughan F. R. Jones. Index for subfactors. *Invent. Math.*, 72(1):1–25, 1983. MR696688 DOI:10.1007/BF01389127.
- [Jon03] Vaughan F. R. Jones. Quadratic tangles in planar algebras, 2003. [arXiv:1007.1158](#).
- [JS91] André Joyal and Ross Street. The geometry of tensor calculus. I. *Adv. Math.*, 88(1):55–112, 1991. MR1113284.
- [KO02] Alexander Kirillov, Jr. and Viktor Ostrik. On a q -analogue of the McKay correspondence and the ADE classification of \mathfrak{sl}_2 conformal field theories. *Adv. Math.*, 171(2):183–227, 2002. MR1936496 [arXiv:math.QA/0101219](#) DOI:10.1006/aima.2002.2072.
- [KW93] David Kazhdan and Hans Wenzl. Reconstructing monoidal categories. In *I. M. Gel'fand Seminar*, volume 16 of *Adv. Soviet Math.*, pages 111–136. Amer. Math. Soc., Providence, RI, 1993. MR1237835 (preview at google books).
- [Lon94] Roberto Longo. A duality for Hopf algebras and for subfactors. I. *Comm. Math. Phys.*, 159(1):133–150, 1994. MR1257245.

- [LR97] R. Longo and J. E. Roberts. A theory of dimension. *K -Theory*, 11(2):103–159, 1997. MR1444286.
- [MPS] Scott Morrison, Emily Peters, and Noah Snyder. Knot polynomial identities and quantum group coincidences, from the d_{2n} planar algebra. forthcoming, will be available at <http://tqft.net/identities>.
- [Müg03] Michael Müger. From subfactors to categories and topology. I. Frobenius algebras in and Morita equivalence of tensor categories. *J. Pure Appl. Algebra*, 180(1-2):81–157, 2003. MR1966524 DOI:10.1016/S0022-4049(02)00247-5 arXiv:math.CT/0111204.
- [Ocn88] Adrian Ocneanu. Quantized groups, string algebras and Galois theory for algebras. In *Operator algebras and applications, Vol. 2*, volume 136 of *London Math. Soc. Lecture Note Ser.*, pages 119–172. Cambridge Univ. Press, Cambridge, 1988. MR996454.
- [Ocn99] Adrian Ocneanu. Paths on coxeter diagrams: from platonic solids and singularities to minimal models and subfactors’. In B. V. Rajarama Bhat, George A. Elliott, and Peter A. Fillmore, editors, *Lectures on operator theory*, volume 13 of *Fields Institute Monographs*, page Part 5. American Mathematical Society, Providence, RI, 1999. MR1743202.
- [Ocn01] Adrian Ocneanu. Operator algebras, topology and subgroups of quantum symmetry—construction of subgroups of quantum groups. In *Taniguchi Conference on Mathematics Nara '98*, volume 31 of *Adv. Stud. Pure Math.*, pages 235–263. Math. Soc. Japan, Tokyo, 2001. MR1865095.
- [Ocn02] Adrian Ocneanu. The classification of subgroups of quantum $SU(N)$. In *Quantum symmetries in theoretical physics and mathematics (Bariloche, 2000)*, volume 294 of *Contemp. Math.*, pages 133–159. Amer. Math. Soc., Providence, RI, 2002. MR1907188.
- [Ost03] Victor Ostrik. Module categories, weak Hopf algebras and modular invariants. *Transform. Groups*, 8(2):177–206, 2003. MR1976459 arXiv:0111139.
- [Pen71] Roger Penrose. Applications of negative dimensional tensors. In *Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969)*, pages 221–244. Academic Press, London, 1971.
- [PP86] Mihai Pimsner and Sorin Popa. Entropy and index for subfactors. *Ann. Sci. École Norm. Sup. (4)*, 19(1):57–106, 1986.
- [RT91] Nicolai Reshetikhin and Vladimir G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.*, 103(3):547–597, 1991. MR1091619 euclid.cmp/1104180037.
- [TW05] Imre Tuba and Hans Wenzl. On braided tensor categories of type BCD . *J. Reine Angew. Math.*, 581:31–69, 2005. MR2132671 DOI:10.1515/crll.2005.2005.581.31 arXiv:math.QA/0301142.
- [Yam03] Shigeru Yamagami. C^* -tensor categories and free product bimodules. *J. Funct. Anal.*, 197(2):323–346, 2003. MR1960417, arXiv:math/9910135.